Constructions*

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Preface: A Story

Once upon a time a young Czech logician, enjoying his stay in England, refused to come back into his country where in the meantime the armies of the Great Brother prepared a re-education of his compatriots. He accepted an offer presented to him by a Professor of Philosophy in Dunedin and went together with his family to New Zealand. In Dunedin he worked hard and after some years he became Professor of Logic there. His extremely sharp mind made him well known not only in Dunedin and not only in New Zealand. He became something like famous especially after he criticised Sir Karl Popper and proved that the concept of verisimilitude, as defined by the latter, involves a contradiction. He did not appreciate this fame, since he considered this criticism as a not very important by-product of his work. And he was right. His work resulted in working out a system of (philosophical) logic, known as *Transparent* (i.e., not respecting 'indirect' or 'oblique' or 'opaque' contexts) *Intensional Logic* (TIL). In the eyes of the world, he was not successful in that none of the stars on the logical sky has accepted or systematically commented his work (the exceptions are rare, and the whole system of TIL has been never accepted by any of the stars). The reasons are varied and mostly they are not important with respect to what this Preface intends to say. There is one point, however, which *is* important and which inspired the authors to writing the present study. The point is that although Tichý's criticism of 'Frege's Thesis', i.e., of his claim that expressions denote actual objects instead of what has been connected with the given expression by linguistic *fiat*, has not been evaluated and appreciated enough, the main obstacle that prevents the contemporary logicians and semanticists from understanding (let alone accepting) TIL consists in refusing to understand the notion *construction*, introduced by Tichý by the seventies and handled by him within a ramified hierarchy of types in his *The Foundations of Frege's Logic* (De Gruyter 1988).

It is a difficult task to find out why this concept has not been accepted with a shout of joy; it helps us in solving some puzzles offered by logicians and makes it possible to analyse, for example, *attitudes* (be it propositional or notional attitudes). Besides, without accepting the concept of constructions you can hardly expect that Frege's hypothesis that between an expression and the object denoted by it there is a *Sinn* that determines the way to the object denoted could be satisfactorily interpreted. It is not as if this 'sense' were easily interpreted as 'intension', i.e., as a function from possible worlds (and times). Intensions are functions, i.e., set-theoretical objects, whereas *sense* should not be reducible to set-theoretical objects. As Zalta says:

> Although sets may be useful for describing certain structural relationships, they are not the kind of thing that would help us to understand the nature of presentation. There is nothing about a set in virtue of which it may be said to present something to us.
>
> [Zalta 1988, 183]

It would not be fair to say that only Tichý began to see that Fregean senses cannot be construed as some set-theoretical entities. The idea of '*hyperintensionality*' can be dated already with Carnap's *Meaning and Necessity*, and explicitly, as the idea of *structured meaning*, can be found in Cresswell's work. (We will have some remarks to this in A.4.) And independently of Tichý the French computer scientist Girard (see [Girard 1990]) does not doubt that Fregean senses cannot be intensions and that they have to be structured. All the same, most of the contemporary semanticists and logicians do their work assuming that what is structured are linguistic expressions, and that a kind of Tarskian denotational semantics is all what there is.

Therefore, we decided – after a vast number of discussions here and abroad – to write a systematic introduction to Tichý's conception of constructions. We proceed as follows: In *Chapter A* (Introduction) we try to suggest some motivation, partly supported by some historical background, and to formulate a global characterisation of Tichý's theory. *Chapter B* (Simple Types) characterises the area of ('first order') objects and defines the 'objectual base' over which the types of order 1 are defined. Most of the philosophical background of the theory is explained in this chapter. Constructions are defined in *Chapter C.* The bare definition is accompanied by important comments. *Chapter D* (Ramified Hierarchy) justifies the transition to higher types and reproduces Tichý's inductive definition. To gain the ability of manipulating constructions is the goal to be achieved with help of *Chapter E* (Exercises); finally, application to the problem of analysing propositional / notional attitudes, *de re* / *de dicto* suppositions are topics of *Chapter F* (Applications).

A. Introduction

1. Logical form, its semantic status

Sainsbury (in his [Sainsbury 1991]), when analysing the distinction between grammatical and logical form, compares two notions of logical form: according to the first one, logical form is ”a sentence in some favoured artificial language”; according to the second, the phrase ‘logical form’ ”signifies not a formalisation into some possibly alien language, but the intrinsic logical and semantic properties of the sentence”. Here we would like to stress that the first of the two notions is connected with some unanswered questions, the most important of which is: if we identify logical form with a sentence of an artificial language, then what could prevent us from asking *what is the logical form of this sentence?* (An artificial language *is* a language, so one would expect that its sentences should possess their logical form as well.) In other words, we could ask about the logical form of the logical form …, briefly, *ad infinitum* (not getting any answer, of course). Thus looking for the (?) logical form of a sentence we rather search for the ”intrinsic logical and semantic properties of the sentence”.

Now what are the essential features that distinguish logical form from grammatical form? There are some illuminating points in Sainsbury’s analysis. Summing up: the main question Sainsbury deals with could be formulated as follows: *If the logical form of a sentence should represent its truth conditions in a perspicuous and systematic way (that preserves compositionality), then how could we explain the cases where an argument in a natural language is valid but not formally valid, as we can see when the sentences of the argument are replaced by their logical forms?*

Any attempt at answering this question has to answer another question: which expressions of a natural language have to be interpreted as *logical constants*, and what is the role played by the latter? This question is important, since for a form to be logical means that the only relevant constants should be the logical constants; as Sainsbury says:

> Logic, and thus formalisation, must have no truck with non-logical constants.
>
> [Sainsbury 1991, 302]

(True, this principle is not as clear as it seems to be: see Bolzano’s theory of entailment. For the time being, let us set this problem aside. See, however, an excellent analysis of this problem in [Etchemendy 1999]). As Sainsbury says on p. 312: ”A language fit for logic [this also means: fit as representing logical form – P.M.] will have no other than logical constants.”

One of the criteria handled by Sainsbury is especially interesting from our viewpoint: we claim that it is unacceptable, and the reason is closely connected with the approach characteristic of the theory of constructions. The criterion is (*ibidem,* 314):

> [a] logical constant is an expression whose meaning is wholly determined by rules of proof.

First, as Sainsbury rightly sees, 'meaning' cannot be identified with truth conditions. But then, of course, the criterion is vague until a definition of meaning is given. But second, what are the 'rules of proof'? Which property does distinguish an arbitrary rule as a 'rule of proof'? For example, why is the Modus Ponens rule a rule of proof unlike, say, the Prior's 'rules'

$$\frac{X}{\therefore X * Y} \qquad \frac{X * Y}{\therefore Y} \qquad ?$$

Unless we are intuitionists or want to be 'pure syntacticists' we have to admit that such a distinguishing property is a *semantic* property. But then the above criterion does not say anything interesting, because the semantically definable property *correctness* or *validity* is dependent just on the meanings of the logical constants, and, indeed, these meanings are "wholly determined by rules of proof". A nice circle.

Anyway, let us suppose that a list of logical constants is somehow available. Is then the problem of logical form solved?

Sainsbury makes some comments to the theory of 'proxy semantics' (Davidson) consisting in translating English sentences to a proxy language Q* which arises from the language of 1[st] order predicate logic, when extra-logical constants are substituted for by English expressions that enjoy the fixed interpretation given by the semantics of English. Sainsbury rightly sees that this solution is not satisfactory, "for we wish to allow logical form proposals to provide a conduit for discoveries about the semantic structure of natural languages". Is there a criterion that would help us to decide which 'proxy language' is the most adequate one?

All such attempts share one feature: the solution is seen in finding some appropriate *artificial language.* Some heretic words about this trend can be found in [Tichý 1994]:

> Modern logic is best described as linguistics without tears. Our declared interest is language. But we have found that the language we actually speak is too complicated and that life is too short. So instead of studying real languages we investigate toy languages invented by ourselves for our own convenience. It is a bit as if zoologists, dismayed by the complexity of real ducks and bears, decided to study plastic ducks and stuffed teddy bears instead. (p.7)

(See also [Tichý 1994a].)

Moreover, the expressions of the respective artificial language *qua* expressions do also possess a logical form. The above-mentioned *regressus ad infinitum* is unavoidable.

Tichý, after he had criticised Chomsky and Montague, began to go his own way. It is essentially distinct from the way characteristic of recent work in the semantics of natural language. Such a typical analyst's concern

> [i]s to defend his favourite toy language; he wants to demonstrate that for any semantic trick one can pull in natural English…a similar trick can be pulled in his toy language. He is not interested in how those tricks are pulled in English. He does not see his task as that of deciphering the code of the natural language. *He is a formaliser, not a code-cracker.*
>
> (*Ibidem,* p.11; emphasis mine. P.M.)

Tichý's way is incompatible with the standard way. This is best seen from his formulation:

> Dragging another *language* into the picture is just an unnecessary complication.
>
> (*Ibidem.*)

So we see that if the logical form is defined in terms of an artificial language, there is no hope to capture the meaning of a natural language expression; this meaning is encoded by an expression of a language **L** in a way specific for **L**, but translatability of languages shows that what is language specific is the way of encoding the meaning: the meaning itself is language neutral, or, if you like, 'international'. Thus, even though there are various ways to the meaning, meaning itself cannot be captured in terms of a specific 'toy language'. This is why *constructions* cannot be defined as artificial expressions.

2. Structured syntax, non-structured semantics?

Let *A* be a natural language expression. *A* serves to *denote* an extra-linguistic object, say, **A**, and to achieve this 'goal' it *expresses* what Frege called its *sense* (or: *meaning*). (The sense of *A* can – with Church – be called a *concept* of **A**. See [Church 1956], [Materna 1998].) Besides, if *A* is an empirical expression, then we can and should distinguish between *denotation* and *reference;* this claim can be justified as follows: let us suppose (as we do) that any rational explication of the Fregean category *sense* (meaning) has to assume that sense *unambiguously* determines the denotation. Thus what is denoted by *A* (i.e., the object **A**) cannot be *co-determined* by something that is beyond the 'competence' of sense, namely, by empirical facts. Now if it seems (as it may seem in the case of empirical expressions) that the denotation somehow depends on the state of the world (for example, that the denotation of an empirical sentence is a truth-value, as if 'varying' dependently on the state of the world at a given time point), then we can define a function whose values would be the seemingly varying values which we originally called ‘denotation’ and whose arguments are the pairs <W,T>, i.e., <possible world, time point>. Calling this function ‘denotation’ we get the required independence of empirical facts (no function is dependent on its arguments, only its value is). So **A** – as a function from possible worlds and time points – is an intension in the sense of Possible-World-Semantics (PWS).

(This terminology slightly differs from Tichý’s; see [Materna 1998].) The reference can be thus construed as being the value (if any) of the intension **A** in W at T, or, alternatively, as being this value in the actual world ‘now’.

To adduce some examples, consider the English expressions a) *nine,* b) *cat,* c) *the morning star.*

As for a), what is denoted here, is the number 9, whatever numbers may be. The sense of a) will be explained later.

As for b), omitting again the sense, what is denoted, is a kind of intension called *property (of individuals).* So the property *to be a cat* is denoted by b). What does b) refer to? It seems that it should be something like *the class of actual cats*. (As we will see in B.3, properties of individuals are in TIL (but in many other systems as well) construed as functions that associate every possible world and time with a class of individuals.)

Finally, consider c). To understand c) we have to interpret *morning star* as a disguised description, something like *the clearest celestial body in the morning sky*. The sense of c) is somehow connected with this description. (This point is very vague, indeed, but, it will be clarified later, and, intuitively, we feel that such a sense could be a way, how the denotation is ‘given’, as Frege’s intention surely was.) Now what is *denoted* by c) is something like a *condition* to be reached by an individual to play the role (to ‘occupy the office’, as Tichý has it) of the morning star. But this can be again modelled as an intension, this time associating the W-T pairs with at most one individual. The contingent (not semantically determined) fact that this condition is fulfilled by Venus in the real world now means that we would call Venus, the individual object, *the reference* of c) (viz. in the real world now).

So far so good. Now suppose that *A* is a complex expression. Let our examples be:

d) 7 + 5,

e) *The highest mountain.*

A serious problem begins to arise already with d): For whereas 7 denotes the number **7,** 5 denotes the number **5**, and + denotes the function known as addition, the whole expression seems to denote the number **12**. Now what is the *semantic* explanation of the fact that the three objects denoted by 7, 5, and + unambiguously determine the resulting number **12**? The pseudo-explanation (to be found, e.g., in [Sluga 1986]) consists in stating that what guarantees this transition from the denotations of particular components of the expression to the denotation of the whole expression is the *grammatical structure* of the expression.

The best criticism of this view I know can be found in [Tichý 1988, 7]:

> If the term '(2.2) – 3' is not diagrammatic of anything, in other words, if the numbers and functions mentioned in the term do not themselves combine into any whole, then the term is the only thing which holds them together. The numbers and functions hang from it like Christmas decorations from a branch. The term, the linguistic expression, thus becomes more than a way of *referring* to independently specifiable subject matter: it becomes *constitutive* of it. An arithmetical discovery must, on this approach, be construed as a finding *about* a linguistic expression.

Thus if the only structure guaranteeing the birth of the *denotatum* from what the particular components of the expression denote is the *syntactic* structure of this expression, a counterintuitive consequence is that *semantics itself* cannot concern some extra-linguistic structures. The denotational semantics, which determines denotations of expressions, is all what there is, and, moreover, to analyse expressions to get their denotations is a matter of linguistic analyses (of particular languages, which, let us not forget, use to possess various grammatical structures).

It was Frege, who first introduced a 'third level' of analysis and suggested that between an expression and the object denoted there is the "mode of presentation" of the object, i.e., sense. Now for Frege it was obvious that this sense has to be an extra-linguistic abstract entity; unfortunately he made no attempt at defining this entity. Thus many logicians accepted the view that sense could be intension, i.e., a function from possible worlds (and times). Surprisingly, they did not take into account the fact that the first place in [Frege 1892] where Frege speaks about sense does not admit of such an interpretation, since his example is from the non-empirical, mathematical area (the example with medians of a triangle). In such examples (see also our d)) we cannot apply the notion of intension: there is no need of possible worlds in mathematics, where every claim holds either in every or in no possible world.

Using our d) instead of Frege's example we can ask: if the result (the number **12**) is determined *via* an extra-linguistic entity called sense, then every particular subexpression of d) must be connected with some sense as well. Then, however, it would be *semantics* what would determine the result, and one can expect that this extra-linguistic sense will be structured, since the sense of ‘7 + 5’ is surely distinct from the sense of, e.g., ’14 – 2’ but the result (the *denotatum*) is the same. Hence *there should be some connection between the linguistic structure and the structure of the sense, and this connection should be distinct from identity.*

An independent argument for the distinct character of linguistic (grammatical) and logical structure follows from the simple fact that when *B* is a correct translation of *A* and the grammars of the respective languages are (often essentially) distinct, it is the sameness of sense of both expressions what enables us to translate *A* to *B*. (This is, what the ‘deflationists’ like Horwich do not take into account.)

These considerations can be applied to our example e). Here both *the highest* and *mountain* can be said to denote (abstract) objects. We will see later that the former can be construed as a function that associates with a given world and time a function which associates any class C of individuals with that one (if any) which is the highest one in C. The latter denotes the property of individuals *being a mountain,* i.e., again a function (intension) that associates with a given world W and time T the class of individuals that are mountains in W at T. Now again, having these two *denotata* we have to combine them to give birth to the ‘individual office’ which is occupied in every world W and time T by at most one individual, viz. with that one which satisfies the condition *to be the highest mountain.* This intension (‘office’, ‘role’) is what is denoted by the expression e), whereas Mount Everest, the real mountain, is the reference of it in the real (‘actual’) world now, being the value of the above function in the actual world now. And again we can expect that the *denotatum* (unlike the reference) is determined by the sense of e), which should be determined by the senses of *the highest* and *mountain*. The reference is not determined by the sense nor by the denotation, for the factor that co-determines it is the state of the world at the given time point.

3. A lesson from Bolzano

Avoiding any anachronism in interpreting Bolzano we all the same can claim that his *theory of concepts* in [Bolzano 1837] contains the idea of structuredness of meaning (sense). His concepts, as a kind of *Vorstellungen an sich*, are non-mental objective abstract entities, and they can be – indeed, in terms of current semantics – construed as (potential!) meanings (senses) of expressions. Now Bolzano’s definition of the *content of a concept* (*Begriffsinhalt*) is in principle distinct from the traditional one. According to the traditional definition, the content of a concept C was the collection of concepts $C_1,\ldots,C_n$ such that the extension of C was identical with the intersection of the extensions of $C_1,\ldots,C_n$. (These C_i were *Merkmale* (perhaps *marks*) of C.)

This definition presupposed two points: a) Every concept is a general concept (*universale*) and b) applying the concept C is equivalent to the *conjunction* of applications of $C_1,\ldots,C_n$. Both points were abandoned by Bolzano.

As for a), Bolzano – unlike, e.g., I. Kant – recognised not only general concepts (*Gemeinvorstellungen/-Begriffe*) but also individual concepts (*Einzelvorstellungen/-Begriffe*) – see §68 of [Bolzano 1837] – and, of course, empty concepts. An example of a concept that would not be recognised as a concept by traditional doctrine is 3^5 (§56). And in the same paragraph Bolzano proclaims a conception of concept (for us also: of sense) that is incompatible with the traditional "conjunctive" conception; he says:

> Da unter diesem Inhalte nur die *Summe* der Bestandtheile, aus denen die Vorstellung (i.e., also a concept – P.M.) bestehet, nicht aber die *Art*, *wie* diese Theile untereinander verbunden sind, verstanden wird: so wird durch diese Angabe ihres Inhaltes eine Vorstellung noch nicht ganz bestimmt, sondern es können aus einerlei gegebenen Inhalte zuweilen zwei und mehr verschiedene Vorstellungen hervorgehen.
>
> (Since this content [viz. of a notion / concept - P. M.] is construed as a mere *sum* of the components of which the notion consists, not however as a *way* in which these parts are combined, declaring the content of a notion does not yet fully determine this notion: there can sometimes arise two and more distinct notions from one single content.)

What is especially important is the fact (see point b)) that "the way in which the components [of the concept] are interconnected" is fully general: there is not only 'conjunction of applications' but any way how the particular components combine to result in the whole concept. To exploit Bolzano's famous example, consider the concept A MAN WHO KNOWS ALL EUROPEAN LANGUAGES (ein Mensch, der alle europäischen Sprachen versteht, §120). Such concepts as MAN, KNOW, EUROPEAN, LANGUAGE,… *are components* of the former concept for Bolzano, but they surely *are not components* of it for the traditional doctrine: the reason is that we cannot say that "a man who knows all European languages" is a man, is knowing, is European, is a language,… . Therefore the above concept *is decomposable* for Bolzano and *is not decomposable* for the traditional doctrine. In order to see the consequences of this for our claim about Bolzano's theory, let us analyze the simpler example given by Bolzano *ibidem*. Bolzano says that the content (*Inhalt*) of the concept [given by the expression] 3^5 is the same as the content of the concept 5^3. A contemporary analysis would offer (where exp is the function defined by $\lambda xy\, x^y$) the following common 'content': {exp, 3, 5}, where the members are concepts connected with the respective expressions. The concept, however, connected with 3^5 consists in *applying* the function exp to the pair <3,5>, whereas the second concept consists in *applying* the same function to the pair <5,3>. These roles of the particular members of the content are not involved in the bare set {exp, 3, 5} and their distinct functioning in both cases is what distinguishes the respective concepts. (In this simple case it seems as if Cresswell's [1985] identification of structures with tuples would be sufficient, but later we will see that it is not the case.)

Speaking about meanings (senses) instead of concepts (which is fair enough, according to [Church 1956]) we can say that for Bolzano the extra-linguistic entity meaning is (or can be) structured and, moreover, that *this structuredness cannot be reduced to merely possessing parts*: the set containing particular components of meaning does possess parts in a sense (even this could be doubted, as we will show later) but the character of the structure called meaning (concept) is of a higher type.

Bolzano's wonderful anticipation of problems from the second half of the 20th century, as well as his anticipation of their possible solution (although formulated only in terms then accessible) deserves our admiration. It is what we called *A lesson from Bolzano.*

4. Carnap, Cresswell

In modern times it was Gottlob Frege who first met the problem of propositional attitudes. In his [1892] he recognised that his chosen method of analysis would come into conflict with the principle of compositionality, if applied without any modification to 'belief sentences' (so the problem of 'oblique contexts' came into being). Frege's attempt at solving this problem is well-known: it is a typically contextualistic attempt, leading to as many 'senses' in an expression as there are 'levels of embedding' of this expression. The second milestone in the development of *constructions* is, of course, Rudolf Carnap, who in his [1947, §30] reacted to Frege's attempt by a keen criticism, showing that the resulting 'pluralism of senses' is untenable (and not clear enough). In §13 he formulates the problem and states that his dichotomy of expressions (either extensional, or intensional) cannot be applied to the case of belief sentences: having a belief sentence of the form, say, "John believes that …" we have to admit that "the whole belief-sentence is neither extensional nor intensional with respect to the subsentence … ". In order to capture the semantically relevant features of the belief contexts Carnap introduces his notion of *intensional isomorphism*. Alonzo Church [1954] formulates some clever objections to this Carnap's notion, to which Carnap replies in Supplement C of his book, where, among other things, he tries to reinterpret belief sentences by basing them on an attitude to *sentences.* The point that is most important for us is that Church considers propositional attitudes to be links between individuals and *propositions*, whereas Carnap thinks that this is only one of the two options. We can see that Church has to explain how to fit together the set-theoretical character of propositions (as intensions in Carnap's or PW-semanticists' sense) with the fact that identity of propositions is not sufficient to preserve equivalence of belief sentences; on the other hand, Carnap's attempts are too closely connected with the surface form of particular languages and are, therefore, exposed to whatever objections that can be raised against *sententialism.* (See, e.g., [Schiffer 1987], [Tichý 1988, 12-14].)

Already in 1975 (and, of course, e.g. in 1985) Cresswell takes up again the Carnapian problem and – mentioning [Lewis 1972] – introduces the term *hyperintensionality*. Cresswell's idea is to define languages that preserve compositionality and make it possible to formulate "a truth-conditional semantics, which yet accommodates propositional attitudes" ([1975, 33]). This attempt results in his well-known 'tuple theory', a criticism of which can be found in [Tichý 1994a]. Cresswell's solution is not radical enough. Cresswell defines hyperintensional contexts as "contexts that do not respect logical equivalence" and he believes that this can be explained as soon as 'hyperintensional functors' are defined whose arguments are intensions (in the PW-semantics sense); he assumes that "all meanings are intensions". Thus his approach is a set-theoretical approach, since functions are set-theoretical objects, and introducing functors/functions whose arguments are intensions (also functions!) does not change the set-theoretical character of this approach. Besides, tuples are also set-theoretical entities representable by functions from natural numbers.

So Cresswell, just as Carnap, does see the problem of non-structured objects vs. structured expressions, but his 'structured meanings' are not structured. Commenting Cresswell's analysis of subtracting 5 from 9 as < -, 9, 5>, Tichý says:

> A convention is needed which *interprets* the triple as a proxy for the construction of applying the first component…to the other two as arguments. … The triple merely enumerates the objects of which the construction is composed; it does not combine those objects into the construction. It is an Ersatz of the construction. [1994a, p.78].

5. Functional approach. λ-calculi.

The theory of constructions is based on a *functional viewpoint*; this means that the most fundamental pre-theoretical notion underlying the theory is the notion of function, rather than that of set or relation. (True, functions are also sets/relations, but we will see that constructions can be easily defined as combining functions; under a functional approach we can enjoy the operation of applying function to arguments, which has no counterpart in the relational approach. See, however, later.) In this respect the theory of constructions is anti-Russellian. This point is explained in [Tichý 1988, 68-70]; Russell's ramified hierarchy knows only sets/relations. Therefore, how a function adheres logically to its argument, can be explained by Russell only if there is a fact which *instantiates* the function, in other words: if there is a *fact* constituted by the combination of a function and its argument, in other words, if the values of the function are truth-values. For then we can, of course, say that, e.g., the combination of the function *Odd* with the argument 3 constitutes the fact that 3 is odd. It is, however, impossible to explain, within Russell's system, how a 'non-propositional function' can combine with its argument: for example, the function *Suc*(cessor) combines with 3 to give 4, but there is no 'fact' which would explain this result. Similarly, as Tichý states, combining propositional functions with such arguments which make the resulting 'proposition' false (e.g., *Odd* with argument 2) cannot be explained in a Russellian way: again, there is no corresponding fact here.

A very expressive tool for dealing with functions has been described by A.Church [1940]. His λ-calculus, originally typeless and important for investigating the problems of computability (λ-definability!), has been connected with type-theoretical hierarchy (to avoid paradoxes). As Manzano [1997] says:

> The offshot of this was fantastic, since added to the formalising capacity of the lambda language was the naturalness of type representation. (226)

From our viewpoint, the most important contribution of the typed λ-calculus to the functional approach in logic consists in the following principle:

Analyses based on the functional approach need two fundamental operations: a) 'creating' a function by 'abstracting', b) applying the given function to an argument.

This principle makes it possible to equip truth-functional connectives and quantifiers with a self-contained semantics rather than to call them 'improper symbols' and interpret only 'contexts'. To adduce an example, where the standard way of interpreting connectives has the form

A • B is true iff … ,

where • is a binary connective, we can associate • with a well-defined truth-function and the context A • B replace by the operation of applying this function to the given pair of truth-values. The expressive power of the operations a) and b) above is considerable.

The theory of constructions is inspired by this principle, and, therefore, by the typed λ-calculus. This is not to say that it is only some modification of the latter. In the next section we will show some important distinctions.

6. The core of Tichý's theory of constructions

The main point where Tichý's theory of constructions (TTC), see, e.g., [Tichý 1986,1988, 1995] differs from the λ-calculus is that

λ-terms, unlike constructions, are linguistic expressions.

One could say, simplifying a little, that λ-terms are – from the viewpoint of TTC – *names of constructions.* This is, however, not the way how λ-calculus interprets λ-terms: consider an example of an interpreted λ-term (x ranging over, say, real numbers):

$\lambda x\ (x > 0)$.

This term is interpreted as a name of the set of positive (real) numbers. But this is not what this expression of an artificial language means. If we wanted to build up semantics of the λ-calculus (which we do not want), then the set of positive numbers would correspond to Frege's *Bedeutung,* to the *denotatum*; the above interpretation would be part of *denotational semantics*, which is correct but which – as we can read in [Girard 1990] – "misses the essential point". This point would correspond to Fregean *sense*, or, as we are used to say, *meaning*. The meaning of the above λ-term should be the abstract procedure that is (in Fregean parlance) *a mode of presentation* (*Art des Gegebenseins*) of the set of positive numbers. As such it would be, naturally, a non-linguistic entity, which, on the other hand, would share with the λ-term something that the denotatum, the class of positive numbers, cannot share with it: namely, a structure that can be derived from (albeit not being identical with) the structure of the term. The 'result', the class of positive numbers is just a class: it does not contain components that would correspond to the components of the term.

Indeed, this distinction between the object that has been constructed and the way it has been constructed is the starting point of Tichý's theory of constructions. Its fundamental importance of this distinguishing can be illustrated by two points.

First, it is clear (and can be easily proved) that any object can be constructed in infinitely many ways. To adduce a simple example, consider any truth-functional object, i.e., a function which can be represented by a matrix ('table') where on the left side there are m-tuples of truth-values and on the right side truth-values. Such a function can be constructed by infinitely many constructions whose linguistic 'codes' are known as well-formed formulae of the truth-functional calculus. We can ask: what does such a wff semantically represent? I think that there is no other answer than to say that such a wff represents one of the infinitely many constructions of the respective function. For one example consider the truth-function conjunction. Its table looks as follows:

p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

There are, however, infinitely many constructions of this table. One of them:

$\neg (\neg p \vee \neg q)$,

which is, of course, another wff, but which 'encodes' some abstract procedure that leads to the same truth-function as $\wedge$.

Remark: This view, viz. the construal of expressions as codes for abstract procedures (let us speak already now about *constructions*) enables us to resolve the *puzzle of defining tautologies*: Consider how to define tautologies.

First option: *A (say, truth-functional) tautology is a wff whose value is Truth on any valuation.*

Second option: *A tautology is a function that associates every tuple of truth-values with Truth.*

Now the puzzle consists in showing that both options are strongly counterintuitive:

a) The first option is counterintuitive. For consider following 'tautologies':

$p \supset (q \supset p)$,

CpCpq.

No doubt, these are two distinct wffs, so they are two distinct tautologies. Yet we feel that to say this is patently absurd; we should, instead, say that what we see are *two distinct notations of one and the same tautology.* After all, we are never taught that every notational system has special tautologies of its own.

b) The second option is counterintuitive. For consider following 'tautologies':

$p \supset (q \supset p)$,

$\neg p \vee p \vee q$.

According to the second option both wffs symbolise one and the same tautology. Yet we are always taught the banal truth that there are infinitely many tautologies (for any *n*-tuple of truth-values). Here we would get just one (for any *n*-tuple of truth-values).

Now considering the way the constructions have been characterised we know the way out. Take the case a). Both wffs encode one and the same sequence of 'intellectual steps' (the Polish notation does it more naturally): the truth function *implication* is applied to the pair <truth-value, the result of applying the same truth function to the pair <truth-value, truth-value' >>. For each of the 4 possible pairs of truth-values we get Truth. Choosing any formulation of this procedure we can see that *both the wffs are expressing the same abstract procedure*.

Now take the case b). This time the 'intellectual steps' describable w.r.t. the first and second wff are distinct. Both procedures construct one and the same function ('TRUE') but the ways they do it are not the same.

Thus we can say that

i) distinct notations may express one and the same construction (*language independence of constructions*) and

ii) whereas functions are 'coarse-grained', *constructions are nearly as fine-grained as linguistic expressions*.

Second, we simply cannot analyse some expressions semantically unless we apply the notion of construction. A classical example: Considering a sentence like

Charles calculates 7 + 5

we have to reflect upon the type-theoretical character of the relation *calculate*. (Even if we chose some typeless system, some similar reflection would be necessary.) Calculating is clearly a binary empirical relation which links individuals with something. Now which options do we have when determining this something?

It looks like as if we had again two options.

i) Calculating links individuals with (say) arithmetical expressions.

ii) Calculating links individuals with numbers.

We now show that both options are absurd.

Ad i): Calculating is not a linguistic but an arithmetical activity. Consider the following variant of our sentence:

Charles calculates seven plus five.

Our report about Charles' activity will surely be the same. Or observe the translation

Karl zählt sieben plus fünf.

The fact that calculating is *not* a linguistic activity can be demonstrated by observing the following sentence:

Charles writes "seven plus five".

Writing is obviously a linguistic activity, therefore the German translation

Karl schreibt "sieben plus fünf".

would be wrong: the correct translation must be

Karl schreibt "seven plus five".

So the first option is untenable.

Ad ii): The second option is not less absurd. If the number-result of calculating were what calculating would link the individual to, then the analysis of our sentence would be the same as the analysis of the sentence, say,

Charles calculates 3 . 4,

but then the same analysis would be associated with two distinct non-equivalent sentences.

Is there any other option?

If constructions (to be defined!) are what can be conceived of as meanings (or senses) of expressions, then we can say that calculating is a relation linking individuals with *meanings* of the expressions following the verb *calculate* in the respective sentence. **Again, constructions are *language independent*** (case i) **) and *fine-grained*** (case ii)).

The phrase *being (nearly as) fine-grained (as the respective expression)* means that *the structure of an expression mirrors the structure of the respective construction in a language-specific way.*

B. Simple Types

1. Base

Constructions are defined so that they are *universal* as for which objects they construct. Yet there have to be some objects, and the realm of objects, the 'ontology', has to be given. The way this is done in TIL is now described.

The lowest level of objects, 'first order' objects, involves just such objects that are not constructions or classes, relations, etc., with constructions as members. The most universal 1st order ontology is inspired by ordinary natural language: which kinds of object are we able to talk about in such a language?

There is one unifying criterion of building up such ontology, if our approach is – as it actually is – a functional one. A 1st order, 'simple', functional hierarchy of types can be defined, based on the idea that all abstract (1st order) objects can be modelled as functions over some collections of 'basic atomic objects'. These collections (*simple atomic types*) are chosen with the purpose to make it possible to model all abstract objects we would like to talk about in a natural language.

Remark: One could wonder why we take *abstract* objects only into account. With the possible (but not absolutely clear) exception of proper names we really speak about abstract objects. The most illustrative examples are empirical descriptions. We already analysed the expression *the highest mountain.* The concrete object, Mt Everest, is not what we name by this expression: we have seen that it is what could be called an 'individual role' or, as Tichý has it, 'individual office', which takes its values from the particular worlds+times. As a function it is an abstract object. The abstract character of the mathematical objects is beyond any doubt, and empirical expressions always denote intensions (i.e., abstract objects) only, never their (contingent) value in the actual world. This is in full harmony with what Gödel should have said to Hao Wang [1992]:

> [S]ymbols only help us to fix and remember abstract things: in order to identify concepts, we associate them with certain symbols.

Concrete (i.e., spatially and temporally localisable) objects can be given (maybe exclusively) by ostension, often accompanied by some demonstrative.

Now try to imagine which kinds of object we talk about using a natural language.

a) No discourse is thinkable without distinguishing **T**(rue) from **F**(alse). Therefore, the collection of the two *truth-values* is surely an indispensable type. This type will be denoted ο (omíkron).
b) TIL assumes that among the lowest-level objects there have to be simple (i.e. unanalysable) objects, the 'lowest-level bearers of properties'. These objects are called *individuals* and the collection of individuals is denoted ι (iota).
c) *Time points* play their role as abstract objects we often speak about (*via* some time units) and which stand behind our use of tenses. Since time can be modelled as continuum, the type-collection of time points serves also as the type-collection of *real numbers*. This type, playing the above double role, is denoted τ (tau).
d) The fourth atomic type is the most controversial one. Without it we could, however, make no logically tractable distinction between i) modalities and ii) empirical and non-empirical expressions. It is the collection of *possible worlds* denoted ω (ómega).

Comments:

Ad a): TIL is in a sense a *classical logic*, for it admits only two truth-values. The values that the many-valued logics work with are no *truth*-values; the cases where the presence of truth-value gaps is provable can be easily handled, since TIL defines functions as *partial functions.* If *undefined* were to be conceived of as a 'third value', then Bochvar's logic would satisfy TIL definitions unlike Lukasiewicz's etc.

Ad b) TIL is an anti-essentialist system. Individuals can possess empirical properties but none of them necessarily. There are no 'essential' empirical properties. There are, of course, many open questions here, see, e.g., [Tichý 1983], but this topic cannot be analysed here for our theme is *constructions*, rather than *intensional logic.*

Ad c) Natural languages, at least what we mean by 'ordinary talk', do not take into account sophisticated theories of time. So no applications of, say, theory of relativity can be expected here.

Ad d) Possible worlds can be pre-theoretically characterised as time sequences of 'sets of compatible facts' (the Wittgensteinian conception). Ascribing an explicit type to them makes TIL a 'two-sorted' (better: many-sorted) logic, whereas the too cautious Montague did not classify possible worlds with his types, preferring the implicit way of writing ($s \rightarrow \alpha$) as types of intensions and associating no extra type to *s*. One of important consequences of the 'Wittgensteinian' conception is that it is impossible to know which of the possible worlds is the real, 'actual' one: such knowledge implies omniscience.

More motivations of this choice of atomic types can be found in the excellent Chapter 11 of [Tichý 1988].

Definition 1 The *objectual base* is the set {ο, ι, τ, ω}.

Now the 1[st] order types are atomic types plus *collections* of partial functions over the objectual base. Inductively:

Definition 2 i) Every member of the objectual base is *a type of order 1.*

ii) Let α, $\beta_1,\ldots,\beta_m$ be *types of order 1.* Then the collection of all partial functions with values in α and arguments (*m*-tuples with members) in $\beta_1,\ldots,\beta_m$, respectively, is *a type of order 1*. It is denoted ($\alpha\beta_1\ldots\beta_m$).

iii) Only what satisfies points i) and ii) is *a type of order 1.*

The 'objectual' character of TIL is perspicuous here: The standard method of defining consists in defining 'type expressions' (which might also be called 'types') and *interpreting* them in a separate definition. Such a 'formal' method yields no advantage whatsoever.

2. A type-theoretical analysis of the members of the ontology

Let us inspect now whether both above definitions make it possible to classify type-theoretically all interesting (abstract) objects we want to speak about. For this purpose let us draw up a (perhaps non-exhaustive but at least) representative list of such objects. Where α is a type (here: of order 1 later, any type) we will speak about α-objects as members of the type α.

A list of objects to be represented type-theoretically:

a) Classes of α-objects;

b) Relations-in-extension (i.e., non-empirical relations) of β_1-,…,β_m-objects;

c) Properties of α-objects;

d) Relations-in-intension (i.e., empirical relations);

e) Propositions;

f) Magnitudes;

g) Truth functions;

h) Quantifiers over α;

i) Identity over α;

j) Singulariser over α.

Notice that all these categories are meant in the objectual sense: their members are members of ontology, they are not linguistic expressions. This may be terminologically strange in the case of quantifiers, but what is strange is at most the terminology here: do not forget that according to Definition 2 all abstract objects that are not members of atomic types are functions, i.e., mappings, and we will easily show that even the (ontological counterparts of) traditional quantifiers are functions. Similarly, there are ontological counterparts of descriptive operators, but here I have chosen (along with Tichý) another term: singulariser.

Ad a): Let A be a class of α-objects. Logically, there is no distinction between a class and its characteristic function. So the type of A will be ($o\alpha$), which will be symbolised by A/ ($o\alpha$). For example, if A is a class of individuals, its type will be ($o\iota$), if it is a class of numbers, its type will be ($o\tau$); if A is a class of classes of individuals, its type will be (o ($o\iota$)), etc.

Classes of α-objects are ($o\alpha$)-objects.

Ad b): *m*-ary relations(-in-extension) are classes of *m*-tuples. Again, we can identify them with characteristic functions, so that we immediately have:

Relations-in-extension of β_1-,...,β_m-objects are ($o\beta_1...\beta_m$)-objects.

Ad c): Properties (here meant exclusively as empirical properties): (Empirical) properties can be best modelled as functions. Since we can distinguish between *temporal* and *modal variability* of the 'population' of a property ([Tichý 1988], [Materna 1998]), we can 'compose' the required function of the function *chronology* (of classes), type (($o\alpha$)τ), which for each time point returns at most one ($o\alpha$)-object, and the function which for any possible world returns at most one such chronology. So we get the type ((($o\alpha$)τ)ω). Indeed, take any property of individuals (so that α is ι), e.g., to be green. The temporal variability can be illustrated by the fact that given a definite possible world the population of green individuals changes with time. The modal variability is given by the fact that even at a definite time point, the fact that an individual is (is not) green is *not a necessary fact: it could be in another way.* Instead to say that the given individual could *not* be green we say that there are possible worlds where it is not green. Thus the property of being green can be modelled as a function that for a given possible world returns the chronology of classes of individuals characteristic of this world.

Properties of α*-objects are* ((($o\alpha$)τ)ω)*-objects.*

Ad d): By analogy we can immediately claim:

Relations-in-intension of β_1-,...,β_m*-objects are* ((($o\beta_1...\beta_m$)τ)ω)*-objects.*

Abbreviation: Let α be any type. The type (($\alpha\tau$)ω) will be written $\alpha_{\tau\omega}$.

Ad e): The term *proposition* possesses many meanings. In the Possible-World-Semantics (PWS), to which TIL belongs, a proposition is what can be – dependently on world-time – true or false (or, in specified cases, without any truth-value at all). So propositions are, in PWS, non-structured (non-Russellian) propositions (see [King 1997], where Tichý's constructions are just what the author needed without being aware thereof). They are, indeed, functions the values of which depend on the possible world and the respective time point. Obviously, they can be modelled as functions that in the given possible world return its chronology of truth-values.

Propositions are $o_{\tau\omega}$*-objects.*

Ad f): The problem of magnitudes has been made famous in connection with the Quinean 'number of planets (of our Solar system)' example. The prejudice according to which the expression *the number of planets (of our Solar system)* denoted the number 9 led to the well known 'puzzle' with necessity. Actually, this expression denotes a magnitude: given a world we get a chronology of numbers; the respective concept is an empirical one.

Magnitudes are $\tau_{\tau\omega}$*-objects.*

Ad g): Truth functions are simply functions from tuples of truth-values to truth-values. Let C be an *m*-ary truth function. Then C, applied to an *m*-tuple of truth-values, returns (at most) a truth-value.

m-ary truth functions are $(o\ldots o)$*-objects, where the number of* o*'s is m + 1.*

Ad h): Let $\forall$, $\exists$ be the (ontological counterpart of the) universal and the existential quantifier over a type α, respectively. $\forall$ is a function which, when applied to a class of α-objects, returns **T** or **F** according to whether this class is or is not identical with the whole type α. $\exists$ is a function which, applied to a class of α-objects, returns **T** or **F** according to whether this class is or is not non-empty.

Quantifiers over α *are* $(o(o\alpha))$*-objects.*

Ad i): Identity over any type is the set of all pairs <i,j> such that i is the same object as j.

Identity over α *is an* $(o\alpha\alpha)$*-object.*

Ad j): A function called here *singulariser* is defined as follows: Applied to a class it returns nothing (is undefined) if this class is empty or if it contains more than one member. Otherwise, i.e., if the class is a singleton, singulariser returns as the value the only member of that class. Singularisers are what semantically underlie definite descriptions (which need not be eliminated, unless we share – as we do not - Russell's aversion to partiality).

Singulariser over α *is an* $(\alpha(o\alpha))$*-object.*

We could continue, but the above list is sufficiently convincing. We state that the system based on the two above definitions allows us to handle logically objects of the ontology, at least by defining the above type-theoretical classification.

3. Intensions, extensions

Each 1st order object belongs to one of two important classes: it is either an *intension* or an *extension.* The following definitions will make it clear that what we mean by these terms differs from that use which is characteristic of those writers who speak about *intension of, extension of.* So we will never say that an expression *has, possesses* etc. an intension on one hand, and an extension on the other hand. Instead, any expression either denotes an intension, or it denotes an extension. As a rule, the former expressions are empirical, unlike the latter. Let the object **a** be the denotation of the empirical expression *A*. What is meant by 'extension of *A*' (e.g., in Montagovian works) is obviously the value of the intension **a** in the actual world, i.e., something what is unattainable semantically and has to be explored empirically. So far about the mentioned distinction in using terms *intension, extension* here and, e.g., in Montague.

Definition 3 Let α be any type (of order 1, later: any type). To be *an intension* is to be an $\alpha_{\tau\omega}$-object. An object **a** is *a non-trivial intension* iff it is an intension and there are at least two possible worlds W, W' in which the values of **a** are distinct. The 1st order objects that are not intensions are called *extensions.*

Remark: In a sense, trivial intensions 'behave' like extensions, but they are distinct in that they are functions from possible worlds (and times). An innocuous ambiguity arises, of course, when we ascribe types to objects. Thus every number can be a τ-object as well as a $\tau_{\tau\omega}$-object; in the latter case we get a trivial intension (a constant function whose value is in every possible world one and the same chronology, which again is a constant function whose value is one and the same number in every time point).

We have briefly described the 'intensional environment' of the world of constructions. Many further details, technicalities as well as philosophically relevant points, can be found in Tichý's book and his articles (see Bibliography) and in [Materna 1998]. The import of the suggested ontology becomes clear as soon as TIL is applied to analysing natural language expressions (see also parts E and F of the present study).

C. Constructions

Constructions, as structured entities, will be inductively defined. The so generated hierarchy has to stop somewhere, i.e., there have to be some most simple constructions which make it possible to realise a kind of 'interface' between constructions and objects. In TIL we can define two kinds of such simple constructions: *variables* and *trivialisation.*

Remarks:

1) Essentially, the following definition reproduces Tichý´s definition from his [1988]; we have only left out his *execution* and *double execution*, since it seems that the contexts referring to them can be equivalently replaced by contexts lacking such references.
2) The definition of constructions is Definition 4. Since it is inductive and proceeds stepwise it has 'separate' parts 4a), 4b), 4c) and 4d).
3) The entity defined is *construction.* Therefore, this term will be always in italics, as it is usual in definitions. Besides, each particular kind of construction possesses its name. These names are in bold-faced letters. This 'double emphasising' cannot lead to any misunderstanding.

1. Contact with objects:

a) Variables

The standard conception of variables has it that variables are *letters, characters,* i.e., *linguistic expressions.* If constructions are to be non-linguistics entities (as it should be clear from the above considerations), then, of course, variables cannot be letters.

What follows is an informal characteristics of variables as a kind of construction. (See [Tichý 1988].)

Let α be a type containing at least two members. Then there are countably infinite many infinite sequences of the members of α ('sequences over α'). Also, we have countably infinite many *variables over* α at our disposal. Given one of the sequences, say, S_i and the *k*-th variable, say, x_k, over α we say that x_k *constructs the k-th member of* S_i. Thus we see that what a variable constructs depends on which sequence of the given type is given. So we define total functions each of which associates every variable with just one sequence of α-objects. (Thus we can say that each of these functions selects just one such sequence.) We call such functions *valuations.* Now if there are more types at our disposal (actually, in our case there are infinitely many types, see Def. 2), then each valuation associates each type with just one sequence over the respective type. We have an infinite array

$X^1{}_1, X^1{}_2, \dots,$
$X^2{}_1, X^2{}_2, \dots,$
.
.
.

where the $X^i{}_j$ is the *j*-th sequence over the *i*-th type. In effect, a valuation associates each variable over a type with just one object of that type. Thereby what is constructed by the given variable is determined: the *k*-th variable constructs the *k*-th object in the sequence given by the valuation. To indicate this dependence on a valuation we will say that variables *v-construct* objects, where *v* is the parameter of valuations. So variables are incomplete constructions; they – as well as the constructions that contain them – do not construct objects, they *v*-construct objects.

The letters usually used as variables ($x, y, z, x_1, x_2, \dots,$ any arbitrarily chosen letters) are according to this conception *names of variables.* Technically, variables can be handled in the same way as they are handled in the usual ('linguistic', 'Tarskian') way.

Example: Let a construction contain a numerical variable x_3 and a variable t_2 ranging over binary truth functions (i.e., over (οοο)-objects). Let a valuation *v* associate with τ the sequence 0, 0, 2, 4, … , and with (οοο) the sequence disjunction, conjunction, implication, … . Then x_3 *v*-constructs 2 and t_2 *v*-constructs conjunction.

Definition 4a) Variables are *constructions.*

b) Trivialisation

One way how to 'feed' constructions by objects has been shown in Definition 4a). There is another way.

Definition 4b) Let X be any object/construction. ^{0}X is a *construction* called **trivialisation.** It constructs just X without any change.

It could seem that trivialisation is as much trivial as not necessary at all. Yet there are at least two points that justify introducing this kind of construction. First, the remaining constructions, no more mediating the contact with objects, can be 'fed' by objects only *via* variables or just trivialisation, since they produce constructions from 'subconstructions' which are, of course, no proper objects. Second, we will see later that *mentioning* (vs. *using*) constructions is realisable only *via* trivialisation or some 'essentially similar' kind of construction.

Already these two points show that the import of trivialisation will be clear later. Now only simple examples.

${}^{0}2$ constructs number 2. ^{0}mountain constructs the property *(being a) mountain* (an $(o\iota)_{\tau\omega}$-object); let x_1 be (the first) numerical variable, then ${}^{0}x_1$ constructs this very variable (constructs, not *v*-constructs!). This last example shows that variables under trivialisation are not free, they cannot range over the given type, they are simply given, a sort of mentioning, not using them.

2. Applying function to argument:

c) Composition

Evaluating a simple arithmetic expression, say,

2 + 3,

we can see that semantically we can distinguish between what a Fregean would call 'denotation' (or, in a more 'stylish' way, 'reference') – and this is obviously the number 5 – and something what would correspond rather to the Fregean 'sense' (in English mostly 'meaning'). Recapitulating that sense/meaning should determine the denotation and that one and the same denotation can be determined by more (actually, infinite number of) senses, we can consider another expression whose sense also determines the number 5. Let it be

7 – 2.

Here we have a classical Fregean example: two distinct expressions and one and the same denotation. Could we find out which kind of entity would play the role of the Fregean sense?

(An answer to such a question – a very similar to ours – can be found in the introductory chapter of [Girard 1990].)

We will exploit our functional approach. In both cases we have three simple subexpressions, in first case '2', '+', '3', in the second '7', '-', '2'. Each of them denotes an object. '+' and '-' denote certain arithmetic functions, '2', '3', '7' denote numbers. Now to get the number 5 we have to *apply* the function denoted by '+' to the pair of numbers denoted by '2', '3', respectively (in the case of the first example) or (in the second case) to *apply* the function denoted by '-' to the pair of numbers denoted by '7', '2', respectively.

Can the set {+, 2, 3} or the set {-, 7, 2} be what we could call 'sense/meaning' of the given expression? Surely not (see the quotation from Zalta in Preface). Neither the ordered lists of those elements, say, <+, 2, 3> or <+ <2,3>> etc. can (see the section **F**). What is missing is the operation of applying function to its arguments. Maybe something like <Apply, +, <2,3> > would be what we are after? But again, what is missing is the operation of applying the function denoted by the ‘Apply’ to the pair <+, <2,3>>.

A ‘linguistic solution’ might be offered: it is the grammar of the (here: arithmetic) language what guarantees the transition of the above set-theoretical entities to the denotation. We have already quoted Tichý´s criticism of this proposal (see **A.2**).

Let us list in more details ‘intellectual steps’ that lead to the number 5 in our examples.

1. Identify the function +.	1’. Identify the function – .
2. Identify the number **2.**	2’. Identify the number **7**.
3. Identify the number **3.**	3’ Identify the number **2**.
4. Apply the function + to **2**, **3**, respectively.	4’ Apply the function – to **7**, **2**, respectively.

Remark: A slight modification of TIL types (see Def.2) introduces another type, ‘*tuple type*’, and – as a consequence – two kinds of construction (‘*tuple*’ and *‘projection*’): see C4. Accepting this modification we would have to insert a step before 4 and 4’ above, consisting in creating pairs of the respective numbers.

Now the steps 1 – 3 (1’ – 3’) can be represented by trivialisations. It is the step 4 (4’) what illustrates the need of another kind of construction, consisting in applying function to arguments.

Definition 4 c) Let X, X_1, …, X_m be *constructions* which *v*-construct $(\alpha\beta_1\ldots\beta_m)$-, β_1-,…,β_m-objects, respectively. $[XX_1\ldots X_m]$ is a *construction* called **composition**. If the function **F** *v*-constructed by X is not defined on the tuple of objects $\mathbf{b}_1,\ldots,\mathbf{b}_m$ *v*-constructed by $X_1,\ldots,X_m$, then $[XX_1\ldots X_m]$ is *v-improper,* i.e., it does not *v*-construct anything. Otherwise it *v*-constructs the value of **F** on $\mathbf{b}_1,\ldots,\mathbf{b}_m$.

Remark: Notice that a composition is *v*-improper also if any of X, X_1, ..., X_m is *v*-improper.

Returning to our simple examples, both arithmetic expressions can be conceived of as encoding the sequence of intellectual steps described above. In other words, they encode following constructions:

$$[{}^0{+}\ {}^0 2\ {}^0 3],$$
$$[{}^0{-}\ {}^0 7\ {}^0 2].$$

To see that constructions (and, therefore, compositions) are no linguistic expressions it is sufficient to realise that, for example, no composition contains brackets; the composition is the *abstract procedure itself*, and our convention given by Definition 4 c) concerns only the notation. If we decided that composition would be written in another way, e.g., with normal parentheses, nothing would change as for the composition itself.

Another point important for understanding constructions: notice that the specific operation characteristic of composition is *application of function to arguments.* There is nothing here what would be specific for a particular language. To come from an expression to the respective application-procedure is another problem for English than, e.g., for Chinese, but this distinction is given by the fact that the grammar of English is (essentially) distinct from the grammar of Chinese; the underlying construction is, however, the same.

Remark: A dangerous misunderstanding could be caused by our term 'intellectual steps' (used also in [Tichý 1988]). An impression could arise as if constructions were *mental* or *mind-dependent entities* (some kind of Brouwerian constructions). The term is meant either as a metaphor, or as concerning not the construction but a *use of construction.*

Two simple examples illustrating composition:

i) $[^0\!: {}^0 5\ x_1]$, x_1 ranging over τ, is a v-improper construction for all valuations that associate x_1 with 0.

ii) $[^0\!: x_1\ {}^0 0]$ is v-improper for all valuations.

Other examples will follow, especially in Chapters **E** and **F**.

3. Getting function by abstraction:

d) Closure

The second kind of construction inspired by λ-calculus guarantees that functions can be not only applied to arguments but also 'created'. True, already now we can get any function *via* variables and trivialisations; since variables can range over any type, a valuation can cause that a function is constructed as a value of the variable of the respective type; or a trivialisation will construct a function like in the case ^{0}sin. Much more important is however the way based on abstraction. Verbally, the procedure can be described as follows: Let C be a construction which (v)-constructs members of a type α. A new construction can be constructed over C so that – for some $m > 0$ – this new construction (v)-constructs a function that for any m-tuple of values of some types $\beta_1, \ldots, \beta_m$ (not necessarily distinct) returns at most one value which is the result of replacing the (possibly occurring) variables contained in C by the respective values of the given m-tuple and letting C construct the value after this replacement. The type of the resulting function is, of course, $(\alpha\beta_1\ldots\beta_m)$ (see Definition 2). A precise definition follows:

Definition 4d) Let $x_1,\ldots,x_m$ be pairwise distinct variables ranging over $\beta_1,\ldots,\beta_m$, respectively (β_i, β_j not necessarily distinct types for $i \neq j$) and let X be a *construction* v-constructing α-objects. Then $[\lambda x_1\ldots x_m \text{ X}]$ is a *construction* called **closure.** It v-constructs the following function **F**: let <$\mathbf{b}_1,\ldots,\mathbf{b}_m$> be a tuple of β_1-,…,β_m-objects, respectively, and v' be a valuation that associates x_i with $\mathbf{b}_i$ and is identical with v otherwise. Then **F** is undefined on <$\mathbf{b}_1,\ldots,\mathbf{b}_m$> if X is v'-improper (see Definition 4c)); otherwise, the value of **F** on <$\mathbf{b}_1,\ldots,\mathbf{b}_m$> is what is v'-constructed by X.

Remarks:

1) The outmost brackets surrounding the expression for closure may be omitted.
2) Only composition can be v-improper: every closure v-constructs a funstion. Compare the composition $[^0\!: x\ ^0 0]$ with the closure $\lambda x\ [^0\!: x\ ^0 0]$. The former is v-improper for any valuation, whereas the latter constructs a function that is undefined on every argument.
3) Now we would like to strongly re-emphasise the non-linguistic objective character of constructions. They are abstract procedures, and the way we denote them is only a convention that determines the character of the respective procedure. Thus we can claim that $\lambda x\ [^0\!> x\ ^0 0]$ constructs the class of positive numbers (for x ranging over real numbers) but we cannot say that it contains λ or brackets. (We can, of course, say that '$\lambda x\ [^0\!> x\ ^0 0]$' contains - as a notation - λ and brackets.) So, we have to emphasise that no closure contains brackets or 'λ': both are only our means how to fix ('encode') the '(abstract) activity' done by the respective construction.
4) The construction $\lambda x\ [^0\!> x\ ^0 0]$ contains *one* occurrence of x. The occurrence of 'x' in 'λx' is an occurrence of the sign for *x in our notation*; it serves to indicate the character of the procedure.

Examples:

a) *Some specific closures*.

Let x be a variable ranging over some unspecified type α. Then $\lambda x\ x$ constructs the *identical function* from α onto α.

Let **X** be any object, e.g., the number 2. Then $\lambda x\ ^0\mathbf{X}$, e.g., $\lambda x\ ^0 2$, where x ranges over any type, e.g., over τ, is an instance of a construction constructing a *constant function*.

b) Arithmetical constructions.

Let *x, y* be numerical variables. (We start with arithmetic, i.e., the simplest examples.) Comparing the following constructions could be instructive.

$[^0\!> x\ ^0 2]$ *v*-constructs **T** or **F** according to whether *x v*-constructs a number greater than 2 or not.

$[^0\!> x\ y]$ *v*-constructs **T** or **F** according to whether *x v*-constructs a number greater than that *v*-constructed by *y*.

$\lambda x\ [^0\!> x\ ^0 2]$ constructs the function that associates numbers with **T** or **F** according to whether the number is or is not greater than 2. But this is just to say that what is constructed is *the class of numbers greater than 2.*

$\lambda x\ [^0\!> x\ y]$ *v*-constructs the class of numbers greater than the number *v*-constructed by *y*.

$\lambda xy\ [^0\!> x\ y]$ constructs the class of such pairs of numbers where the first member of the pair is greater than the second. In other words, it constructs the (numeric) relation >.

$\lambda yx\ [^0\!> y\ x]$ constructs the same relation. Notice that this construction is in a well definable sense equivalent to the preceding one, but is *not identical* with it.

$\lambda yx\ [^0\!> x\ y]$ constructs the class of such pairs of numbers where the second member is greater than the first. In other words, it constructs the (numeric) relation <.

$\lambda xy\ [^0\!> y\ x]$ constructs the relation < . Notice that this construction is in a well definable sense equivalent to the preceding one, but is *not identical* with it.

Now what about

$\lambda x\ [\lambda y\ [^0\!> x\ y]]$? According to Definition 4d), this construction constructs a function which associates every (real) number *n* with what $\lambda y\ [^0\!> x\ y]$ *v'*-constructs for $v'(x) = n$. In other words, the above construction constructs the function that associates every number *n* with the class of those numbers that are less than *n*. For example: applying the function constructed by our construction to the number 2 we get the class of all numbers *k* such that 2 is greater than *k*, in other words, the class of all numbers less than 2. The construction that realises this application is

$$[[\ \lambda x\ [\lambda y\ [^0\!> x\ y]]]\ ^0 2].$$

(Or, omitting the outmost brackets, we can write

$$[\lambda x\ [\lambda y\ [^0\!> x\ y]]]\ ^0 2.)$$

We can see that the construction

$$\lambda y\ [^0\!> ^0 2\ y]$$

does the same job as the preceding one; this fact can be generalised and captured by the constructional counterpart of the well-known β-rule in λ-calculi. A general formulation of this rule in terms of constructions will be given in section D.

What our examples also show is that whether a construction X *v*-constructs or simply constructs depends on whether X does or does not contain free variables. We will define *free variables, λ-bound variables* and *o-bound variables* later, after the ramified hierarchy will have been defined. Now we only state that where X is a trivialisation ^{0}Y, any occurrence of any variable within X is o-bound, and if X is a closure $\lambda x_1 \ldots x_m$ Y, any x_i, $1 \leq i \leq m$, occurring in X is λ-bound unless it is o-bound in Y.

Now we will show how we should construe the construction underlying the 1st order logic expression (*x, y* again numerical variables).

$$\forall x\, \exists y\, (x < y).$$

We know already (see B2) that quantifiers are (ο(οα))- (here (ο(οτ))-) objects. Thus the respective construction is

$$[^0\forall\ [\lambda x\ ^0\exists\ [\lambda y\ [^0< x\ y\ ]]]].$$

Indeed, the class of the numbers *n* (λ*x*) for which the class of those numbers (λ*y*) that are greater than *n* (the final composition) is non-empty (∃) contains all numbers (∀), so that the construction constructs **T**, which is what is meant by the above *expression.*

Notice that if we chose another notation (a Polish one or any other), the expressions would be distinct but the fact that they correspond to ***one and the same way*** *how to achieve the truth-value* ***T*** *is explained just by the fact that the respective construction is the same.*

We can also say that all such notations possess one and the same *meaning* (Fregean *sense*). Moreover, this explanation holds only if constructions are no linguistic entities, no expressions, for otherwise nothing would be explained; only *regressus ad infinitum* would reward our effort. Also, that sharing the same truth-value cannot explain the synonymy of such only notationally distinct expressions is more than obvious.

c) Analysis of empirical expressions. (See B 3.)

Applying our definitions to analysing empirical expressions we should be able to distinguish between empirical and non-empirical expressions. Our criterion could be: *Empirical expressions denote non-trivial intensions*. A justified objection to this criterion would be, however, that it rather determines the way how to analyse an empirical expression as soon as we know that it is one. So let us formulate a *pre-theoretical intuition* that would help us in the required distinguishing. So consider the pairs of English expressions

nine – the number of (major) planets,

prime number – cat,

two is a prime – Warsaw is the capital city of Poland.

We can suppose that our intuition will classify the first members of these pairs with non-empirical and the second members with empirical expressions. Our task is then to justify this intuition.

Let us first compare the members of the first pair. *Nine* is evidently a name of a number. Let such a number be anything (for us it can be a τ-object), it is an abstract object and there is no way how to make the expression *nine* denote another object, e.g., another number. But not only that: we have seen (A 2.) that there can be a distinction between *denotation* and *reference.* This distinction is demonstrable in the case of *the number of (...) planets.* Here what we called *reference* is dependent on the state of the world. To know that it is just now 9 we had to examine this state of the world – we could not 'calculate' this number by only analysing the expression itself. Nevertheless, the expression itself has offered the criterion according to which we could examine the world: given the world-time, the number will be the cardinal number of the class of those objects that possess the property of being a (…) planet in that world-time. This criterion (it is an intension called *magnitude*) is what is denoted by the expression. So to determine which number satisfies the criterion we need to know two things: the denotation (magnitude) and the state of the world at the given time point. Going back to the expression *nine* we can see that no such distinction is present. *We need not know anything about the state of the world at the given time point in order to know which number is denoted* (here = referred to) by *nine.* In general let us speak about the reference as about the *actual object*, if its identification is dependent on our knowledge of the state of the world at the given time point. (The term *knowledge of the state of the world at the given time point* is here used 'softly', i.e., it is not meant as denoting the total knowledge. Otherwise we could be never said to know anything, since we are not omniscient.)

Considering the second pair, we state that the only feature distinguishing it from the first case is that both expressions are common nouns. What kind of object is denoted by *prime number*? Evidently it is a class of numbers (remember, by our approach an (οτ)-object). To know which numbers are and which are not members of this class is not a trivial problem but in principle it is solvable, and the state of the world–time plays absolutely no role in solving it. On the other hand, to say of an object that it is (is not) a cat presupposes some confrontation of the property *being a cat*, as denoted by the expression, with some fragment of the actual world-time. So we have again two distinct things: the denotation – the 'empirical criterion', here: a *property* – and the reference, which is the *class* of the 'actual cats'. (Or perhaps: the *actual class* of cats.)

Finally, compare the members of the last pair. This time they are sentences. The first sentence is best said to denote a truth-value (here it is **T**). No event in the world can change anything on this truth-value, it is again fully independent of empirical facts. Not so in the case of the second sentence. Here we can observe the modal as well as the temporal variability we have spoken about in B2. As for the modal variability, the truth-value of the second sentence (it is just now **T**) is nothing what could be calculated without communicating with reality. Without this 'examination of the world' we can only guess; *it could be otherwise.* As for the temporal variability, even staying in our reality we have to be aware of the fact that there were time points where the sentence was false: *it was otherwise.* Thus whereas the sentence denotes its truth-conditions (we have called them *a proposition*), it only refers to the respective truth-value.

Now it seems that our pre-theoretical characteristics of the distinction between empirical and non-empirical expressions can be formulated as follows:

If no distinction between the denotation and the reference of an expression E can be discovered, then E is a non-empirical expression. Otherwise, E denotes a criterion of identifying 'actual objects' and refers to these objects by means of experience; E is an empirical expression in this case.

A very important point has to be emphasised: what is denoted is always unambiguously determined by the sense (=meaning) of the expression. Therefore, the types of the objects constructed by the sense of empirical expressions are $\alpha_{\tau\omega}$ for α a respective type. The references of such expressions are values of these functions in the actual world, i.e., they are α-objects, and are co-determined by the given world-time.

Remark: The distinction between denotation and reference has been never formulated *expressis verbis* by Tichý but it can be justified by the spirit of TIL.

Our example will be a TIL analysis of the sentence

The number of (...) planets is greater than 7.

(A classical example in the history of intensional/modal logics.)

First, a *type-theoretical analysis* of the sentence and its components must be done.

The number of , let it be N, obviously denotes a function; the latter, applied to a class of (here:) individuals, an $(o\iota)$-object, returns the cardinal number of that class, i.e., a natural number, but unless we want to refine our analysis beside necessity, we can the value of this function construe as a τ-object. (N is evidently undefined on infinite classes.) We have, therefore, N/ τ(οι). (A non-empirical object!)

Planet(s), let it be P, is a name of a property of individuals, so P/ $(οι)_{τω}$.

Greater than, let it be G, is type-theoretically ambiguous ("type-theoretical polymorphism"). In our context it is clear that it denotes a relation(-in-extension) between numbers, so G/ (οττ).

7 denotes the number 7, so 7/ τ.

The number of (...) planets, as an empirical expression, does not denote the number 9 (it only refers to it) but a magnitude, i.e., a $τ_{τω}$-object.

The number of (...) planets is greater than 7, as an empirical sentence, denotes a proposition, i.e., an $ο_{τω}$-object.

Second, *a synthesis*.

The problem can be stated as follows: We have N/ (τ(οι)), P/ $(οι)_{τω}$, G/ (οττ), 7/ τ; we have to find the construction which constructs, first, the magnitude denoted by *the number of planets /* $τ_{τω}$, second, the proposition denoted by the whole sentence/ $ο_{τω}$.

Let us begin with the number of planets. Since this is a $τ_{τω}$-object, it could be constructed by a construction of the form λ*w* [λ*t* X], abbreviated λ*w*λ*t* X, where w ranges over ω, t ranges over τ (we will indicate this by writing w…ω, *t* ...τ; in general, if a construction C *v*-constructs an object of type α, we write C ... α) and X *v*-constructs a number (if any). Further, we can easily see that X has to be the construction (composition) which applies N to a class, i.e., to an (οι)-object. Yet P is not a class: it is a property, and there is nothing like a cardinal number of a property. Hence $[^0N\ ^0P]$ does not work, it is no construction at all. We are, however, after constructing that cardinal number of any *class* that is the value of the *property* P in the given world-time. Let us, therefore apply P to a possible world, obtaining $[^0Pw]$ (i.e., the chronology of classes of planets in *w*) and then apply this result to a time point, obtaining $[[^0Pw]t]$ (i.e., the class of planets in the world *w* at the time point *t*).

Remark: Where X (*v*-)constructs an $α_{τω}$-object, we will write X_{wt} instead of $[[Xw]t]$.

Now $[^0N\ ^0P_{wt}]$ *v*-constructs the number of those individuals which happen to be planets in the given world at the given time point. Finally, the number of planets is constructed by

$$λwλt\ [^0N\ ^0P_{wt}].$$

Now the resulting proposition will be constructed by a construction of the form

$$\lambda w \lambda t\, [{}^0G\ Y\ {}^07],$$

where Y *v*-constructs a number, a τ-object. The only candidate for Y seems to be the construction of the number of planets (see above), but again, instead of τ (which we need), the number of planets is a $\tau_{\tau\omega}$-object. Thus we have to apply this object to the respective world-time. We get

$$\lambda w \lambda t\, [{}^0G\ [\lambda w \lambda t\, [{}^0N\ {}^0P_{wt}]_{wt}]\ {}^07].$$

After executing the composition

$$\lambda w \lambda t\, [{}^0N\ {}^0P_{wt}]_{wt}$$

(see the 'β-rule' in b)) we get

$$[{}^0N\ {}^0P_{wt}]$$

and the final result of our analysis of the above sentence is

$$\lambda w \lambda t\, [{}^0G\ [{}^0N\ {}^0P_{wt}]\ {}^07].$$

We can compare this result with our linguistic intuition. The final construction constructs the proposition that is true in those worlds and times where the cardinal number of the class of individuals which possess the property *being a planet* in the given world-time is greater than 7. But this is how we would construe the truth conditions given (denoted) by the above sentence.

Exercise

(More exercises are to be found in Chapter E.)

Find the construction expressed by the sentence

All swans are white.

(Hint: Do not forget that the quantifiers are – in our case – (ο(οι))-objects!)

4. Other constructions?

The idea of constructions is based on the λ-calculus-like reduction of all operations to *creating functions* and *applying function to arguments.* Adding the necessary stops, i.e., *variables* and *trivialisation*, we get the four kinds of construction as defined in Definition 4. A natural question arises: Can we be content with these four kinds? There are perhaps still further kinds of what we would like to call 'construction'.

Indeed, the author of the idea defines in his [Tichý 1988, 62, 63] two other kinds:

> For any entity X we shall also speak of the *execution* of X and symbolise it as ^{1}X. If X is a construction, ^{1}X is X. … If, on the other hand, X is not a construction then ^{1}X is the (abortive) construction whose starting point is X and which yields nothing. (A non-

construction cannot be executed.) Thus if X is a *v*-improper construction or not a construction at all, ^{1}X is *v*-improper; otherwise it *v*-constructs what is *v*-constructed by X.

…

If what is constructed by X is itself a construction, one can execute X and go on and execute the result. We shall speak of this two-stage construction as the *double execution* of X and symbolise it as ^{2}X. For any entity X, the construction ^{2}X is *v*-improper…if X is not itself a construction, or if it does not *v*-construct a construction, or if it *v*-constructs a *v*-improper construction. Otherwise ^{2}X *v*-constructs what is *v*-constructed by what is *v*-constructed by X.

To adduce some simple examples, let x be a numerical variable, x … τ. The execution ^1x is x, the double execution ^2x is *v*-improper for any *v* (x does not *v*-construct any construction). $^2(^1x)$ is *v*-improper, $^2(^0x)$ *v*-constructs the same as x.

Surprisingly enough, you can hardly find a place in [Tichý 1988] where the two kinds of construction would be relevantly exploited. (True, the double execution is used in Appendix 2 in Lemma 19.2.) We are convinced that they are not indispensable. As for the execution, it serves mainly to the purpose of distinguishing between constructions and non-constructions, which, however, can be achieved independently. As for the double execution, whatever can be done by this kind of construction, can be done *via* a respective function **f** such that given a construction ***c*** and a valuation *v*, **f**, applied to ***c***, returns the object (if any) which is *v*-constructed by ***c***. Thus we do not use the two additional kinds of construction.

This is, of course, not a general answer to the question whether our list of four kinds of construction is exhaustive. Nevertheless, the reduction underlying this choice of constructions is fascinating, and the class of problems solvable by means of these four kinds is remarkable. (By the way, we could consider an opposite problem: could we *omit* some constructions? That such an idea is realistic can be seen when considering Curry's combinators without variables.) The problem is similar to the problem whether our choice of atomic 1st order types is exhaustive (for example: we would need perhaps some 'spatial points' like the 'time points'?).

One attempt at a conceptual enlargement of the theory of constructions, based on an idea of J.Zlatuška [Zlatuška1986] and motivated by some practical purposes in the area of the theory of databases (conceptual modelling) should be mentioned. It can be found, e.g., in [Duží 2000], and the first step consists in adding a new type to Definition 2. In our present notation this type would be defined as follows:

> ii') If $\beta_1,\ldots,\beta_m$ are *types of order one,* then $(\beta_1,\ldots,\beta_m)$, for $m > 1$, called a *tuple type,* is a *type of order 1.*

(Clearly, this type is the Cartesian product of the particular components.) The second step adds two new kinds of construction to Definition 4. The respective two definitions would be formulated as follows:

d') Let $X_1,\ldots,X_m$ be *constructions*. Then $(X_1,\ldots,X_m)$ is a *construction* called (m-)**tuple**. If $X_1,\ldots,X_m$ v-construct objects $\mathbf{X_1},\ldots,\mathbf{X_m}$, respectively, then $(X_1,\ldots,X_m)$ v-constructs the tuple $<\mathbf{X_1},\ldots,\mathbf{X_m}>$. Otherwise it is v-improper.

d'') Let X be a (m-)tuple. Then $X_{(1)}, \ldots, X_{(m)}$ are *constructions* called **projections**. If X v-constructs $<\mathbf{X_1},\ldots,\mathbf{X_m}>$, then $X_{(i)}$ (for $1 \le i \le m$) v-constructs $\mathbf{X_i}$, otherwise it is v-improper.

In this way tuples are rehabilitated as objects *sui generis.* In TIL tuples play their role only indirectly as arguments of functions, but they are never independent objects; so we can speak about elements of classes and construct them, but elements of relations cannot be constructed unless some points like ii'), d'), d'') are added to the respective definitions. This is, however, what we need in databases, since such queries that can be answered only by representing a tuple are frequent. In general, let C be a construction v-constructing truth-values and let $x_1,\ldots,x_m$ be distinct variables ranging over $\beta_1, \ldots, \beta_m$, respectively. Imagine that we want to know *the only* such m-tuple that satisfies C. In the non-extended TIL the respective query would have the form

$$\lambda x_1 \ldots x_m \, C,$$

so that the answer, i.e., what is constructed by the query, would be a relation with just one member. Yet we would prefer to get this member (i.e., this m-tuple) rather than the 'singleton relation'. With the points ii'), d'), d'') we have got the possibility to do it. We have already defined singularisers as functions of the type $(\alpha\,(o\alpha))$. Generalised singularisers have the type $((\beta_1,\ldots,\beta_m)(o\beta_1\ldots\beta_m))$; if the given relation contains just one member, the generalised singulariser returns this very member (m-tuple), otherwise it is undefined. Let S^m be such a singulariser in our case. Then the query would have the form

$$[{}^0S^m\,[\lambda x_1 \ldots x_m \, C]]$$

and the answer would be (in the favourable case) the respective m-tuple.

Remark. When we accept tuples as a kind of construction it could seem that Tichý's criticism of Cresswell would be no more grounded: Cresswell's 'meanings' would be structured. Yet tuples as constructions defined above do not construct what Cresswell intends and what is guaranteed by *composition.* □

Summing up: TIL is surely an open theory, but any attempt to modify it by adding some further types/constructions should be well justified.

C'. Intuitionists' constructions

The term 'construction' itself is not the happiest one: it is connected with many more or less irrelevant connotations. One of them is, however, not unimportant. Intuitionists who attended Tichý's lecture [1995] were deeply interested in how far one could articulate an analogy between constructions in TIL and the intuitionist notion of construction as used in Brouwer's, Heyting's, Sundholm's, P.Martin-Löf's and other intuitionists' or constructivists' work.

Let us therefore offer several remarks concerning this problem. We will exploit [Sundholm 1983] and [Sundholm 1986] because a clear interpretation and analysis contained in both articles make it possible to formulate the present comments in a most concise way.

First, the intuitionist constructions are *proofs*: they prove propositions. (See Kreisel, in [Sundholm 1983, 154]). If so, then the intuitionist notion is too narrow in comparison with the TIL constructions. A proof (a sequence of TIL constructions, well) concerns only sentences, or propositions, whereas the TIL constructions construct objects of any types, ensuring, among other things, that the construction that constructs a proposition can be derived from the constructions that construct the components of such a construction (compositionality).

Second, speaking about the 'meaning of a proposition' (but what is a proposition?) the intuitionists say that the "assertion condition is not propositional"(see [Sundholm 1983, 162]). There is a point where the underlying intuition is shared by TIL. For the TIL constructions of propositions it holds that they are principally distinct from the proposition itself; true, the notion of proposition is precisely introduced in TIL, it is a function/mapping from possible worlds-times to truth-values, but just because of it a propositional construction has to be (as an abstract procedure) distinct from such a function.

Third, Kolmogoroff's interpretation, according to which "propositions express intentions towards constructions", or "pose problems which are solved by carrying out constructions" (see [Sundholm 1983, 159], see perhaps also [Materna 1970]) needs only a slight reformulation to become well compatible with TIL: imagine that a construction itself poses a problem which is solved by carrying out the construction; in the case of empirical constructions, the problem to be solved is given by the outcome of the construction, i.e., by an intension. Its solution is in this case co-determined by the state of the world.

Fourth, Sundholm in [1983, 164] analyses the term 'construction' and shows that it could be interpreted at least in three ways: as "(a) process of construction, (b) object obtained as the result of a process of construction, (c) construction-process as object". If we want to compare the TIL and the intuitionist notion of construction we have to be aware of the fact that the TIL constructions are most likely constructions in the sense c).

Fifth, there is a big problem in the philosophy of intuitionism, which arises as soon as we accept the principle formulated in [Sundholm 1983, 162, "III"]:

> Proofs begin with immediate truths (axioms) which themselves are not justified further by proof…

Accepting this principle together with the intuition according to which

> [f]ailure to understand a meaningful sentence seems parallel to failure to follow, or grasp, a proof

(see [Sundholm 1986, 493]) we are confronted with the problem formulated in [Sundholm 1983, 169]:

> if the meaning of a proposition is explained in terms of proof, can one then only understand proved propositions?

In TIL, where verification of a proposition is distinct from the (constructional) identification of the proposition, no such problem arises.

Sixth, construing constructions as languageless (see, e.g., Brouwer, [Sundholm 1983, 164]), intuitionism could be said to share this idea with TIL, but, alas, the languageless character of intuitionist constructions is compensated by the view that they are *mental.* It should be clear by now that the TIL constructions are abstract rather than mental.

Seventh, the problem of 'empirical constructions', i.e., of constructions that construct intensions like properties or propositions etc., is not taken care of in intuitionism. This is indirectly admitted in [Sundholm 1986, 503], where the donkey-sentences are analysed. Sundholm says:

> [t]he treatment of *atomic* sentences such as 'OWN[*x, y*]' is left intolerably vague in the sketch above and it is an open problem how to remove that vagueness.

Properly speaking, there are two problems mentioned in this quotation: the problem of atomic sentences and the problem of empirical sentences. Both problems, as well as their 'meet' (the problem of analysing empirical atomic sentences), are principally solved in TIL (which does not mean that *everything* is clear by now) but the intuitionist notion of a construction-proof is too specific to be able to elaborate such a general theory of constructions which could compete with TIL in this respect. After all, Sundholm himself writes (*ibidem*):

[o]ne should stress that it is not at all clear that one can export the 'canonical proof-object' conception of meaning outside the confined area of constructive mathematics.

An interesting contribution to a general theory of constructions from the intuitionistic viewpoint offers Fletcher in his [1998]. He characterises constructions as follows (p.51):

...a type of construction is specified by some *atoms* and some *combination rules* of the form 'Given constructions $x_1,...,x_k$ one may form the construction $C(x_1,...,x_k)$, subject to certain conditions on $x_1,...,x_k$ '. A construction, then, is defined recursively as either an atom or $C(x_1,...,x_k)$, where C is a combination rule and $x_1,...,x_k$ are constructions satisfying the conditions for applying C.

This characterisation fits well the constructions in our sense. Fletcher's intuitionistic specification is, of course, distinct. Two points are interesting here:

First, the philosophical distinction consists in the intuitionistic assumption that constructions are *mental* entities. *Second*, Fletcher identifies constructions with „recursive structures interpretable as partial recursive functions" (p.82). (If, however, functions are mappings, so that p.r. functions can be defined as such mappings that can be determined *via* recursive structures, then p.r. functions *as mappings* can be hardly constructions as characterised above.) Here a problem arises with the possibility of generalising this conception to (meanings of) *empirical expressions*: the recursive character of such constructions as

$$\lambda w \lambda t\ C,$$

where C is any construction, is at least dubious. Besides, there would be no constructions available as underlying expressions denoting *non-recursive* functions.

D. Ramified Hierarchy

1. Using and mentioning constructions

There is a distinction between using and mentioning constructions, analogous to but distinct from using and mentioning expressions. When *using* a construction we are interested in what it constructs. When *mentioning* a construction we are interested in the construction itself, we speak about the construction. (See [Duží, Materna 1995] .)

Up to now we are able to use constructions, i.e., we can analyse some expression that is about an object whose type is of order 1. To adduce a simple example, the analysis of the sentence

Two times two equals two plus two

will be

$$[^0\!= [^0\!\times {}^0 2\ {}^0 2][^0\!+ {}^0 2\ {}^0 2]]$$

which can be (sloppily) read as ‚what is constructed by $[^0\!\times \ldots]$ and what is constructed by $[^0\!+ \ldots]$ is the pair of numbers which belongs to the relation constructed by ${}^0\!=$' . We can, however, easily imagine the situation when we have to analyse a sentence like

Two plus two contains the adding operation.

If this sentence possesses a rational reading, then it has to be construed as speaking about the construction underlying the expression *two plus two,* for the number constructed by this construction cannot contain anything, let alone the adding operation.

Remark: Notice that the above sentence says something distinct from what the following sentence says:

The mathematical transcription of the expression 'two plus two' contains the symbol of the adding operation.

Whoever does not see this distinction has not yet grasped the concept of constructions as extra-linguistic entities.

Now we are no more able to analyse the above sentence (preceding the Remark). We should use a construction which constructs another construction (i.e., the *mentioned* construction) but in order to do it we have to ascribe a type to the mentioned construction. Yet whereas we know that the construction $[^0\!+ {}^0 2\ {}^0 2]$ constructs a τ-object (the number 4), we have got no possibility to ascribe a type to the construction itself.

Our situation is a little strange: due to the definition 4b) we are able to construct constructions but no type can be ascribed to the result of such kind of constructing. To see this consider the most simple case: Let x_1 be a numerical variable, i.e., we can write $x_1 \ldots \tau$. That is, the type of what is *v*-constructed by this variable is unambiguously determined by our definitions. Now consider the construction $^0x_1$. Our definitions determine that this construction constructs just the variable x_1. We could write $^0x_1 \ldots$ the type of x_1, but this is what is not determined. Constructions have got no type whatsoever.

Yet we should be able to analyse not only expressions that speak about the 1st order objects. One could ask, which kind of expressions could it be? Maybe only such highly artificial sentences like *Two plus two contains the adding operation,* whose content can be easily circumscribed by speaking about the respective expressions?

At least two answers can be offered.

First, it is not the case as if the above-italicised sentence were equivalent to the 'circumscribing sentence' like that adduced in our last Remark. Let these two sentences be S1, S2, respectively. It can be shown that S1 is true independently of the notational means of a language. Imagine such a mathematical notation where our usual symbol of adding is replaced, say, by a designated letter, e.g., **A**. Then, of course, the sentence S2 is false while S1 remains to be true. (Even if our convention that stipulates the way how to represent constructions were changed, S1 would be true; it is most important to understand this distinction. *Constructions contain functions, not symbols of functions.)*

Second, the cases where constructions that construct constructions are needed for analysis of an expression are by far not always such artificial cases as we have used. In Section F we will see that the frequently used phrases expressing the so-called propositional or notional attitudes cannot be analysed within the 1st order frame.

Therefore, we will extend our type hierarchy so that it were possible to speak about constructions themselves as about objects *sui generis.* The definition that follows is a modified definition from [Tichý 1988].

2. Higher orders

Higher orders will be defined inductively.

First (**T1**), types of order 1 are defined (*via* a reference to Definition 2).

Second (**C*n***), constructions of order *n* are defined (notice that constructions of order *n* are *not* types of order *n*).

Third (**T*n*+1**), the inductive step to types of order n+1 is executed.

Definition 5.

T1

Types of order 1 are types defined in Definition 2.

C*n*

Let α be a *type of order n.*

i) If ξ is a variable ranging over α, then ξ is a *construction of order n.*

ii) Let X be an α-object. Then ^{0}X is a *construction of order n.*

iii) Let C be a composition $[XX_1 \ldots X_m]$ and let n be the highest order such that at least one of X, X_1, …, X_m is a *construction of order n.* Then C is a *construction of order n.*

iv) Let C be a closure $[\lambda x_1 \ldots \lambda x_m\ X]$ and let n be the highest order such that at least one of $x_1, \ldots, x_m$, X is a *construction of order n.* Then C is a *construction of order n.*

v) Only what is defined in i) – iv) is a *construction of order n.*

T*n*+1

Let $*_n$ be the collection of all constructions of order n.

i) $*_n$ is a *type of order n* + 1.

ii) Let n + 1 be the highest order such that at least one of the types α, β_1, …, β_m is a *type of order n* + 1; then $(\alpha\beta_1 \ldots \beta_m)$ (see Definition 2) is a *type of order n* + 1.

iii) Only what is defined in i), ii) is a *type of order n* + 1.

Now a simple example will be useful. Let again x_1 be a numerical variable, i.e., $x_1 \ldots \tau$. Since τ is a type of order 1, x_1 is a construction of order 1 (see Cn i)). Therefore, it is a member of $*_1$ and its type is of order 2 (see Tn+1). Consider now the construction $^0x_1$. Since x_1 is of a type of order 2, $^0x_1$ is a construction of order 2 (see Cn ii)). Therefore, it is a member of $*_2$ and its type is of order 3.

Thus we can write

$$x_1 \ldots \tau,\ x_1/ *_1,$$

$$^0x_1 \ldots *_1,\ ^0x_1/ *_2,$$

etc. etc.

The principle of our notation is now clear: If X is a construction, then by X … α we say that the type of the object (*v*)-*constructed* is α. By X/α we say that the type *of* X is α. If X is an object of a type of order 1, then we can write only X/α; objects whose type is of order 1 (i.e., 1^{st} order objects) are not constructions, so it makes no sense to write X … α. The same holds for any objects that are not constructions.

Definition 6. X is a *higher order object* (HOO) iff X/α and the order of α is greater than 1.

It follows from Definitions 5 and 6 that

a) all constructions are HOOs,

b) not only constructions are HOOs.

The point a) is clear. As for the point b), let the order of at least one of the types α, β_1, …, β_m be greater than 1. Then, according to T*n*+1 ii), any ($\alpha\beta_1\ldots\beta_m$)-object is a HOO without being a construction. As an example, consider the object denoted by the expression *calculate* (assuming that calculating concerns arithmetic constructions). Calculating is obviously an empirical relation that relates individuals with numerical constructions; hence its type is $(o\iota*_1)_{\tau\omega}$ (see B 2). As a relation (-in-intension) it is a function, and *no function is a construction* (by the way, if you are surprised by this last claim, then it means that you still do not understand what kind of object constructions are). On the other hand, confronting this type with T*n*+1 ii) we can see that its order is 2, so calculating is an HOO.

Now we can better see what entities should be called set-theoretical objects.

Definition 7. X is a *set-theoretical object* iff X is a 1^{st} order object or a function (i.e., an ($\alpha\beta_1\ldots\beta_m$)-object for $m > 0$).

As a consequence we can state that *only constructions are not set-theoretical objects.*

Definition 8. Let C be a construction with at least one occurrence of a variable ξ.

i) If C is ξ, then ξ is *free in* C.

ii) If C is ^{0}X, then ξ is o-*bound in* C ("bound by trivialisation in C").

iii) If C is [$XX_1\ldots X_m$], then any occurrence of ξ which is *free,* o-*bound, λ-bound in* X, X_1, …, X_m is *free,* o-*bound, λ-bound,* respectively *in* C.

iv) If C is [$\lambda\xi_1\ldots\xi_m$ X], then any occurrence of ξ in C which is not o-*bound in* X is *λ-bound in* C if ξ is one of $\xi_1,\ldots,\xi_m$; otherwise, ξ is *free in* C.

v) ξ is *free,* o-*bound, λ-bound in* C iff at least one of the occurrences of ξ in C is *free,* o-*bound, λ-bound in* C.

Remark: The above definition implies that one and the same variable may be at the same time free, o-bound and λ-bound in a construction. It is, of course, recommendable to use distinct variables in such cases.

Definition 9. Constructions C, C' are *v-equivalent* iff either both are *v*-improper, or both *v*-construct one and the same object. They are *equivalent* iff they are *v*-equivalent for every valuation *v*.

Remark: Tichý used *congruent* instead of *equivalent.*

Compare the following pairs of constructions ($x_1, x_2 \ldots \tau$):

$$\text{a)} \quad \lambda x_1 [{}^0\!> x_1\, {}^0 0],\ \lambda x_2 [{}^0\!> x_2\, {}^0 0]$$

$$\text{b)} \quad {}^0[\lambda x_1 [{}^0\!> x_1 {}^0 0]],\ {}^0[\lambda x_2 [{}^0\!> x_2\, {}^0 0]]\ .$$

The constructions under a) are equivalent: they both construct the class of positive numbers. Notice that both x_1 and x_2 are λ-bound in them. (Compare the α rule in λ-calculi.)

The constructions under b) are *not* equivalent. They construct distinct (albeit equivalent) constructions. Notice that both $x_1$and x_2 are o-bound in them. (Cf. Definition 8 ii), iv).)

Summing up, a collisionless renaming of λ-bound variables does not change the object constructed. This claim could be called an *objectual counterpart of the α rule.* On the other hand, any renaming of o-bound variables changes the object constructed.

Another example: Following constructions (∧ / (οοο), Pr(ime number) / (οτ), E(ven number) / (οτ), Em(pty class of numbers) / (ο(οτ)), Card(inality) / (τ (οτ)), *x*...τ, *c* … (οτ)) are *not* equivalent:

$$\lambda x\, [{}^0\!\wedge [{}^0\!\wedge [{}^0\mathrm{Pr}\, x]\, [{}^0\!> x\, {}^0 2]]\, [{}^0\mathrm{E}\, x]]$$

$$\lambda c\, [\, {}^0\!\wedge [{}^0\mathrm{Em}\, c]\, [{}^0\!> [{}^0\mathrm{Card}\, c]\, {}^0 1]].$$

Both constructions construct an empty class, but the former constructs the empty class of numbers, whereas the latter constructs the empty class of classes of numbers. Since a class is construed in TIL as being a (characteristic) function, we have infinitely many distinct empty classes; they differ in the arguments of the respective functions.

An 'empirical' example will show that what may seem to be equivalent is not. Let Poland be – for the sake of simplicity only – an individual, so P / ι; T(own) / $(o\iota)_{\tau\omega}$, C(apital city of) / $(\iota\iota)_{\tau\omega}$, L(argest) / $(\iota(o\iota))_{\tau\omega}$, Po(lish) / $((o\iota)(o\iota)_{\tau\omega})_{\tau\omega}$. (See some explanations below.) Compare the following two constructions that should correspond to the expressions *The capital city of Poland* and *The largest Polish town*, respectively (*w* ... ω, *t* ... τ):

$$\lambda w \lambda t\ [{}^0\mathrm{C}_{wt}\ {}^0\mathrm{P}]$$

$$\lambda w \lambda t\ [{}^0\mathrm{L}_{wt}\ [{}^0\mathrm{Po}_{wt}\ {}^0\mathrm{T}]].$$

Even if our analysis were as fine as to capture the semantic dependence of *Poland* and *Polish*, even then we would not get equivalent constructions. The type-theoretical evaluation shows that the type of the objects constructed is the same: $\iota_{\tau\omega}$, i.e., the type of individual office/role (see B1), but the functions of this type differ. There is no way how to get the contingent fact that the value of these functions coincide in the actual world, in other words, that they *happen to* coincide. So the above constructions are *not* equivalent, which is in harmony with our intuition: There is no necessary connection between the fact that X is a capital city of a country and the fact that X is, at the same time, the largest town of that country. The two expressions denote two different offices and just happen to have the same reference.

The above constructions would be equivalent if they constructed (say, the individual) Warsaw, but we see that they cannot construct Warsaw, which is again in harmony with the intuitive principle that *an expression cannot be about something what is not mentioned in it*; no component of the above constructions – as well as no component of the respective expressions – 'mentions' Warsaw. That Warsaw happens to play both constructed roles is confirmed only by a contact with reality.

Remark: We now justify the 'more difficult' types in the above example.

First, consider L / $(\iota(o\iota))_{\tau\omega}$. This is really a function which in every possible world ($_\omega$) associates every time point ($_\tau$) with a function that from any given class of individuals ($(o\iota)$) selects that one (if any) which is the largest one. Which one it is is surely dependent on the world-time: the given individual can possess distinct sizes in distinct world-times.

Second, consider Po / ($(o\iota)(o\iota)_{\tau\omega})_{\tau\omega}$. One would say that *Polish,* as an adjective, denotes simply a property (say, of individuals, i.e., an $(o\iota)_{\tau\omega}$-object). Yet already the Montagovian school discovered that the type to be ascribed to (the objects denoted by) adjectives is not that simple. A property is given only as soon as an adjective is applied to another expression that denotes a property. Take the expressions *big mouse* and *big elephant*. If the type associated with *big* were simply $(o\iota)_{\tau\omega}$, then the analysis of those expressions would be a puzzle. Well, one could say that we should combine two properties, e.g., B(ig) / $(o\iota)_{\tau\omega}$, M(ouse) / $(o\iota)_{\tau\omega}$, and get – using conjunction – the new required property Big_Mouse. But then, by elementary deduction, we would have to claim that a big elephant is big and a big mouse is big as well. So the problem is how to change the type of the adjective so as to avoid this absurd result. Consider therefore the type we have ascribed to Po, i.e., $((o\iota)(o\iota)_{\tau\omega})_{\tau\omega}$, and apply this type in the case of B(ig). *Big*, as an empirical adjective, has to denote an intension. So B will in every world-time associate every *property* with a *class*, which means that no above absurd conclusion can be drawn. In every world-time we will get a new property of being a big mouse or being a big elephant, without deducing that there is some 'independent' property of being big):

$$\lambda w \lambda t\, [{}^0B_{wt}\, {}^0M],\ \lambda w \lambda t\, [{}^0B_{wt}\, {}^0El].$$

This is why Po has got such a 'complicated' type.

Definition 10. Let C be a construction.

i) C is a *subconstruction of* C.

ii) If C is ${}^0X$, then if X is a construction, X is a *subconstruction of* C.

iii) If C is $[XX_1 \ldots X_m]$, then X, $X_1,\ldots,X_m$ are *subconstructions of* C.

iv) If C is $\lambda x_1 \ldots x_m$ X, then X is a *subconstruction of* C.

v) If A is a *subconstruction of* B and B is a *subconstruction of* C, then A is a *subconstruction of* C.

vi) A *subconstruction of* C is only what has been defined in i) – v).

Obviously, being a subconstruction of is reflexive, antisymmetric and transitive. Any construction can be unambiguously decomposed into its subconstructions. (See [Materna, Sgall, Hajičová 1988.] As for the 'bottom-up' direction, there are infinitely many possible expansions (or: syntheses) of arbitrary constructions due to the character of closure (and of trivialisation).

Let X and $C_1,\ldots,C_m$ be constructions. By $X(C_1/x_1,\ldots,C_m/x_m)$ we denote the result of collisionlessly replacing every occurrence of the variable x_i in X by C_i (for $1 \leq i \leq m$).

Claim: *The construction*

$$[[\lambda x_1 \ldots x_m\, X]\, C_1 \ldots C_m]$$

with $x_i \ldots \beta_i$, $C_i \ldots \beta_i$, is equivalent to the construction

$$X(C_1/x_1,\ldots,C_m/x_m).$$

Proof: See [Tichý 1982, 67,68] for the simple hierarchy. Nothing relevant changes when α is a higher order type.

This claim could be called 'the objectual counterpart of the β-rule' known from λ-calculi.

3. Entailment in terms of constructions

The central notion of logic, the *entailment relation,* can be defined as a relation between (sets of) propositions and propositions, so as an $(o\ (o\ (o_{\tau\omega}))\ o_{\tau\omega})$- or $(o\ o_{\tau\omega}\ o_{\tau\omega})$-object; where **A** is a set of propositions or a proposition, and **B** a proposition, **A** is said to *entail* **B** (or **B** follows from **A**) iff **B** is true in all those possible worlds-times where (every member of) **A** is true.

This 'coarse-grained' definition can be replaced by a 'fine-grained' definition:

Definition 11. Let $C_1,\ldots,C_m$, $m \geq 0$, be constructions of order 1, and D a construction of order 1. We say that the set $\{C_1,\ldots,C_m\}$ *entails* D iff D v-constructs **T** for every v such that every C_i ($1 \leq i \leq m$) v-constructs **T**. □

According to Definition 11 entailment is an $(o\ (o*_1)\ *_1)$-object.

A following point is worth a closer examination. Traditionally, whereas provability is considered to be a syntactic notion, entailment is defined as a semantic notion. For ordinary language it can be defined in terms of (structureless) propositions (see above), within the framework of axiomatised systems a set of *formulas* **S** is said to *entail a formula* **B** iff **B** is true in all such *models* in which all the members of **S** are true. In this second case entailment is a relation between formulas defined in terms of their *interpretations* (since models of a set of formulas are simply interpretations in which all members of that set are true). Thus we have some formulas (sentences, if you like) on one hand, their truth-values (given by an interpretation) on the other hand, and entailment relates sentences *via* their interpretations. What is structured are only formulas.

The TIL theory of constructions considers entailment to be an objective relation between *structured non-linguistic objects.* To use the entailment relation no metalanguage is necessary. Moreover, if entailment relates linguistic objects, then a change of notational system means that the entailment relation will contain other members; for the constructional approach, the notation may be arbitrary, the members of the entailment relation are the same for any notational convention. To understand a language as a notational system means to be able to decipher the respective expressions so that what we get are just the constructions.

Indeed, derivatively we can speak about entailment as relating (*sets of) sentences with sentences.*. A sentence (set of sentences) **S** can be said then to entail a sentence **U** iff the construction (set of constructions) underlying **S** entails the construction underlying **U**.

Example

Let Int/ (ο(οτ)) be the class of integers, and Exp/ (τττ) the function $\lambda xy\, x^y$. Let further the variables *x,y,z,u* range over τ. Then the class

$$\{ [^0{=}\ [^0{+}\ [^0\mathrm{Exp}\ x\ u]\ [^0\mathrm{Exp}\ y\ u]]\ [^0\mathrm{Exp}\ z\ u]],\ \ [^0\mathrm{Int}\ x],\ \ [^0\mathrm{Int}\ y],\ [^0\mathrm{Int}\ z],\ [^0{>}\ u\ ^02]\ \}$$

entails

$$[^0\neg\ [^0\mathrm{Int}\ u]]\ .$$

(If, of course, Fermat's Great Hypothesis really holds…)

Yet this 'fine-grained' definition of *entailment* must not be confused with a definition of *provability.* Notice that the class of constructions adduced above entails according to Definition 11 *any construction that constructs* ***T****, i.e., which can be considered to be the sense/meaning of an analytically true sentence.* This is O.K., since analytically true sentences are (derivatively) entailed by any sentence (by the empty class of sentences). But whereas the construction

$$[^0\neg\ [^0\mathrm{Int}\ u]]$$

is a challenge for a mathematician ("Could we prove it from the class above?"), no mathematician will be interested in proving, say,

$$[^0{>}\ [^0\mathrm{Suc}\ x]\ x]$$

or perhaps (*ad absurdum*) the construction that underlies the sentence

Every bachelor is a man

from that class (although both these claims are entailed by it).

In the case of empirical sentences some modification is necessary. Consider the following argument:

The highest mountain is in Asia. MtEverest is the highest mountain. ∴ MtEverest is in Asia.

Type-theoretical analysis: H(ighest) / $(\iota\ (o\iota))_{\tau\omega}$, M(ountain) / $(o\iota)_{\tau\omega}$, (being in)A(sia) / $(o\iota)_{\tau\omega}$, (Mt)E(verest) / ι. Now the particular constructions are, respectively,

$$\lambda w\lambda t\ [^0A_{wt}\ [^0H_{wt}\ ^0M_{wt}]],$$

$$\lambda w\lambda t\ [^0{=}\ ^0E\ [^0H_{wt}\,^0M_{wt}]],$$

$$\lambda w\lambda t\ [^0A_{wt}\ ^0E].$$

How would we demonstrate the correctness of this argument, i.e., the fact that the third construction is entailed by the two other constructions? We can suggest this demonstration as follows:

Let **M** be (the class that is) the value of M in the world W at the time point T. Let **H** be the value of H in W at T. Let **A** be (the class that is the) value of A in W at T. If the first construction constructs **T** in W at T, then (the object/individual that is) the value of **H(M)** is a member of **A**. Similarly, if the second construction constructs **T** in W at T, then **H(M)** is identical with the individual **E** (constructed by E). But then **E** is a member of **A**.

Since this demonstration can be generalised in that it can be repeated without any change for any member of ω, τ, we have proved that the third construction constructs **T** in every world at any time point where the first two constructions construct **T**. This looks like a confirmation of the sound character of Definition 11, but a closer inspection shows that what we did describing the demonstration was that we paradigmatically analysed what happens when a valuation ascribes **T** to the first two constructions. *But valuations concern free variables only*. So we properly speaking ignored the λ-boundness of the variables *w, t*.

Hence to let Definition 11 do its work we would have to omit $\lambda w\lambda t$ in analyses of the empirical premises and conclusions. But since empirical sentences denoting propositions express *closed* constructions, we will use in the empirical case an adjusted definition:

Definition 11':

> Let $C_1,...,C_m$, D ($m \geq 0$) be constructions of propositions $P_1,...,P_m$, P, respectively. We say that the set $\{C_1,...,C_m\}$ *entails* D (or D *follows from* $C_1,...,C_m$) iff $\{P_1,...,P_m\}$ entails P, i.e. P is true in all those possible worlds-times where $P_1,...,P_m$ are.

4. Substitution

Traditionally, if E is a wff of a system (in general, E can be a wff or a term), then the operation **S**, called *substitution*, is defined as follows:

> Let ξ be a variable free in E, and let A be a term. Let A contain variables $\eta_1,\dots,\eta_m$. If a variable η_i , $1 \leq i \leq m$, is in the scope of a quantifier in E, let η_i be replaced by η_i', where η_i' is the first variable not occurring in E or A. Let A' be the result of replacing all the occurrences of η_i in A by η_i' . Let E' be the result of replacing all occurrences of ξ by A'. This operation, leading from E to E', is called *substitution.*

So the 'traditional substitution' is a *linguistic operation*; it concerns expressions of a formal language. The way it is defined serves to the purpose usually called *avoiding collision of variables.* But why should we avoid collision of variables? The reason is, of course, found in semantics: Unless we avoid it, we lose the guarantee that *substitution preserves truth.* Thus we can say that primarily *substitution does not concern expressions as chains of characters but the meanings of those expressions.*

This is what is taken into account in the TIL theory of constructions. Constructions are 'international', i.e., independent of particular languages. Hence the (type-theoretically polymorph) operations $\mathbf{Sub}_k$ are definable in terms of constructions.

First, the concept of (collisionless) substituting is defined as follows (a slightly modified Definition 17.2 from [Tichý 1988]):

Definition 12.

Let C and D be constructions and d a variable. If d is not free in C then *the result of substituting* D *for* d *in* C is C. Assume, therefore, that d is free in C.

1) If C is d then *the result of substituting* D *for* d *in* C is D. …
2) If C is [$\mathrm{XX}_1 \ldots \mathrm{X}_m$] then *the result of substituting* D *for* d *in* C is [$\mathrm{YY}_1 \ldots \mathrm{Y}_m$], where Y, Y_1, …,Y_m are *the results of substituting* D *for* d *in* X, X_1,…,X_m, respectively. Now let C be of the form [$\lambda x_1 \ldots x_m$ Y]; for $1 \le i \le m$, let y_i be x_i if x_i is not free in D, and otherwise the first variable of the same type as x_i, not occurring in C, not free in D, and distinct from $y_1,\ldots,y_{i-1}$; then *the result of substituting* D *for* d *in* C is [$\lambda y_1 \ldots y_m$ Z], where Z is *the result of substituting* D *for* d *in the result of substituting* y_i *for* x_i *in* Y. □

The Compensation Principle is then provable:

Let C be a construction of order 1. Then for any valuation *v* and construction D, if D *v*-constructs *D* then the result of substituting D for d in C *v*-constructs *C* iff C *v'*-constructs *C*, where *v'* associates d with *D* and is otherwise identical with *v*. □

(See [Tichý 1988, 75].)

Proof see *ibidem,* Appendix 1.

Second, the operation $\mathbf{Sub}_k$ is itself a function, type ($*_k *_k *_k *_k$); it is (*ibidem*)

the mapping which takes construction D, variable d, and construction C to the result of substituting D for d in C.

We now reproduce a nice example of exploiting $\mathbf{Sub}_1$. We want to analyse the sentence that claims that *there is such a number x that dividing 3 by x is an improper construction.*

Let Im/ ($\text{o}*_1$) be the class of all improper constructions of order 1. First we show that an immediate attempt at analysis can break down. So let us try as follows:

$$(*) \qquad [{}^0\exists\, [\lambda x\, [{}^0\mathrm{Im}\ {}^0[{}^0{:}\ {}^03\ x]]]]$$

(*) does not work. The closure should produce a class of numbers (whose non-emptiness is claimed due to ∃) *via* letting x range over numbers and associate with **T** those of them for which the improperness of the respective construction holds. Yet no such ranging can take place: the only occurrence of x in (*) is bound (o-bound, see Definition 8).

Remark: (*) is a regular construction, but it is not the result of analysing the above italicised sentence; we can show that the construction

$$\lambda x\ [^0\mathrm{Im}\ ^0[^0\!:\ ^03\ x]]$$

constructs the constant function returning for every (real) number **F** (so that (*) constructs **F**, of course): to be improper is for any construction the same as to be v-improper for every valuation v. Since, however, the construction

$$[^0\!:\ ^03\ x]$$

is not v-improper for every valuation, the construction

$$[^0\mathrm{Im}\ ^0[^0\!:\ ^03\ x]]$$

constructs **F** *independently of* x, which is as it should be, for x is not free here. □

In order to achieve our goal we have to introduce a function Tr/ $(*_1\ \tau)$ which associates every real number with its trivialisation. Then our analysis goes on as follows:

$$[^0\exists\ \lambda x\ [^0\mathrm{Im}\ [^0\mathrm{Sub}_1\ [^0\mathrm{Tr}\ x]\ ^0x\ ^0[^0\!:\ ^03\ x]]]].$$

This time x really ranges, and the class constructed by λx [… is no more constant: it contains 0 (in the sense that the characteristic function returns **T** for the argument 0); indeed, if the trivialisation ^{00} is substituted for x in $[^0\!:\ ^03\ x]$, the resulting construction, i.e.,

$$[^0\!:\ ^03\ ^00],$$

is improper, of course. The (absolutely 'kosher') trick consisted in making x free in $[^0\mathrm{Im}$ [… due to the function Tr. (Notice that *both* other occurrences of x in $[^0\mathrm{Im}$ [… are (o-)bound!)

E. Exercises

Analyse the following sentences, i.e., perform first the type-theoretical analysis of particular simple meaningful components - ‘expression-units’, and afterwards create the construction expressed by a sentence using trivialisations of the simplest objects denoted by the components. By means of the type-theoretical synthesis check the correctness of the analysis.

(S1) There are positive numbers

(S2) Every prime number is odd

(S3) Every prime number greater than 2 is odd

(S4) The Earth is larger than the Moon

(S5) MtEverest is in Asia

(S6) The highest mountain is in Asia

(S7) John is older than Charles

(S8) The king of France is bald

(S9) Charles’s father is smaller than Charles

(S10) Charles’s father is older than his mother

(S11) Charles has some brothers

(S12) Some Charles’s siblings are older than he

(S13) All Charles’s friends are intelligent

(S14) Plato is the most famous teacher of Aristotle

(S15) The king of France does not exist

(S16) The most famous composer is bald

(S17) The richest man is happy

(S18) Rich men are happy

(S19) Some animals are carnivorous

(S20) Honesty is desirable

(S21) Some politicians are honest

Some *solutions*:

(S1) There are positive numbers.

a) Type-theoretical analysis and synthesis

It is a mathematical sentence denoting a truth-value. We can reformulate it as follows:

The class of numbers greater than zero is not empty.

To create a class of numbers we need a λ-abstraction (since we identify classes with their characteristic functions) and a variable x ranging over τ:

$\lambda x\ [{}^{0}{>}\ x\ {}^{0}0]$ ... the class of positive numbers / (ο τ)

Existential quantifier $\exists^{\tau}$ / (ο (οτ)) is the class of non-empty classes (here: of numbers), hence applying it to the above class gives the desired result:

$$[{}^{0}\exists^{\tau}\ \lambda x\ [{}^{0}{>}\ x\ {}^{0}0]]$$

b) Type-theoretical checking

$[{}^{0}\exists^{\tau}$ λx $[{}^{0}{>}$ x ${}^{0}0]]$

(οττ)   τ   τ

ο

(ο(οτ))   (οτ)

ο

(S4) The Earth is larger than the Moon.

a) Type-theoretical analysis

Meaningful components: *the Earth* - E, *is larger* - L, *the Moon* - M

First we have to determine particular kinds of the denoted objects. Here we will ignore the problems connected with the analysis of proper names and take them simply as 'labels' of individuals. Hence *the Earth* and *the Moon* denote individuals. *Is larger than* is obviously a (binary) relation-in-intension between individuals. Thus

types of the denoted objects are: E / ι, M / ι, L / $(ο\ ι\ ι)_{τω}$

b) Synthesis

We have to combine the three simplest components, namely ^0E, ^0M, ^0L, in order to create a construction of a proposition, i.e. $ο_{τω}$ - object. 'Creating a function' in TIL we use closure; thus in case of constructing a proposition we will have: $λwλt$ C, where C *v*-constructs truth values and contains free variables *w*, *t* (if no free variables *w*, *t* occurred in C we would construct a constant function on all possible worlds-times, which is certainly not the case of an empirical sentence, unless it is an analytically true or false one). Applying ^0L to *w* and *t* we get a relation(-in-extension) between individuals. Applying the result to the pair ^0E, ^0M we get a truth value according to whether the individuals Earth and Moon are in the relation of being larger or not. Thus C must be a composition, namely $[^0L_{wt}\ ^0E\ ^0M]$, so that the resulting construction is

$$λwλt\ [^0L_{wt}\ ^0E\ ^0M]$$

c) Type-theoretical checking

$$\begin{array}{lllll} λwλt & [^0L_{wt} & ^0E & ^0M] \\ & (οιι) & ι & ι \\ & & ο & \\ ο_{τω} & & & \end{array}$$

(S6) The highest mountain is in Asia.

a) Type-theoretical analysis

Meaningful components: *the highest* - H, *mountain* - M, *being in Asia* - A

Types of the denoted objects: M / $(\omicron\ \iota)_{\tau\omega}$, A / $(\omicron\ \iota)_{\tau\omega}$, H / $(\iota\ (\omicron\ \iota))_{\tau\omega}$

(M, A are obviously properties of individuals. As for H - it must be an intension which dependently on worlds-times picks up an individual from a class of individuals - the highest one.)

b) Synthesis

The highest mountain denotes certainly an individual office. The sentence claims that the holder of this office in a given world-time belongs to the class of individuals that have (in this world-time) the property A. To construct the office we have to combine ^{0}H, ^{0}M (using again closure):

$\lambda w \lambda t\ [^0H_{wt}\ ^0M_{wt}]$.

Applying this office to *w, t,* we obtain its holder: $[\lambda w \lambda t\ [^0H_{wt}\ ^0M_{wt}]]_{wt}$.

Now the resulting construction will be:

$$\lambda w \lambda t\ [^0A_{wt}\ [\lambda w \lambda t\ [^0H_{wt}\ ^0M_{wt}]]_{wt}\ ],$$

or equivalently after the β-reduction:

$$\lambda w \lambda t\ [^0A_{wt}\ [^0H_{wt}\ ^0M_{wt}]]$$

c) Type-theoretical checking

$$\lambda w \lambda t \quad [^0A_{wt} \quad [\lambda w \lambda t \quad [^0H_{wt} \quad ^0M_{wt}]]\ _{wt}\ ]$$

(ι (οι)   (οι)

ι

$\iota_{\tau\omega}$

(οι)   ι

ο

$\omicron_{\tau\omega}$

(S8) The king of France is bald.

a) Type-theoretical analysis

Meaningful components: *the king* - K, *France* - F*, being bald* - B

Types of the denoted objects: K / $(\iota\ \iota)_{\tau\omega}$, F / ι, B / $(o\ \iota)_{\tau\omega}$

(B is obviously a property of individuals. For the sake of simplicity let us consider F being an individual. As for K - it is an intension that dependently on worlds-times associates an individual with at most one individual - hence it is an empirical function.)

b) Synthesis

The king of France denotes an individual office - the role an individual can play. The sentence claims that the holder of this office in a given world-time (if any) belongs to the class of individuals that have (in this world-time) the property of being bald. To construct the office we have to combine ^{0}K, ^{0}F:

$\lambda w \lambda t\ [^0K_{wt}\ ^0F]$.

Applying this office to *w, t,* we obtain its holder: $[\lambda w \lambda t\ [^0K_{wt}\ ^0F]]_{wt}$.

The resulting construction is:

$$\lambda w \lambda t\ [^0B_{wt}\ [\lambda w \lambda t\ [^0K_{wt}\ ^0F]]_{wt}\ ],$$

or equivalently after the β-reduction:

$$\lambda w \lambda t\ [^0B_{wt}\ [^0K_{wt}\ ^0F]]$$

Note that the proposition constructed by this construction does not have any truth value in the actual world now, for the composition $[\lambda w \lambda t\ [^0K_{wt}\ ^0F]]_{wt}$ is improper (there is no holder of this office in the actual world now).

c) Type-theoretical checking

$$\lambda w \lambda t \quad [^0B_{wt} \quad [\lambda w \lambda t \quad [^0K_{wt} \quad ^0F]]\ _{wt}\ ]$$

$(\iota\ \iota)$ ι

ι

$\iota_{\tau\omega}$

$(o\iota)$ ι

o

$o_{\tau\omega}$

(S10) Charles's father is older than his mother.

a) Type-theoretical analysis

Meaningful components: *Charles* - Ch, *father* - F, *being older* - O, *mother* - M

Types of the denoted objects: Ch / ι, F / $(\iota\ \iota)_{\tau\omega}$, M / $(\iota\ \iota)_{\tau\omega}$, O / $(o\ \iota\ \iota)_{\tau\omega}$

(O is a relation-in-intension between individuals. *Charles* - as a proper name - denotes an individual. F and M are intensions that dependently on worlds-times associate an individual with just one individual - hence they are empirical functions.)

b) Synthesis

We have to construct two individual offices, namely the office of Charles's father and that one of Charles's mother. The sentence claims that the holders of these offices in a given world-time are in the relation(-in-intension) of being older (in this world-time). The two offices are constructed by:

$\lambda w \lambda t\ [{}^0F_{wt}\ {}^0Ch]$, $\lambda w \lambda t\ [{}^0M_{wt}\ {}^0Ch]$.

We have to apply them to *w, t,* to obtain their holders (in *w,* t) who are in the relation of being older, hence the resulting construction is:

$$\lambda w \lambda t\ [{}^0O_{wt}\ [\lambda w \lambda t\ [{}^0F_{wt}\ {}^0Ch]]_{wt}\ [\lambda w \lambda t\ [{}^0M_{wt}\ {}^0Ch]]_{wt}\ ],$$

or equivalently after the β-reduction:

$$\lambda w \lambda t\ [{}^0O_{wt}\ [{}^0F_{wt}\ {}^0Ch]\ [{}^0M_{wt}\ {}^0Ch]\ ]$$

c) Type-theoretical checking:

$\lambda w \lambda t$	$[{}^0O_{wt}$	$[\lambda w \lambda t$	$[{}^0F_{wt}$	${}^0Ch]]_{wt}$	$[\lambda w \lambda t$	$[{}^0M_{wt}$	${}^0Ch]]_{wt}]$
			$(\iota\ \iota)$	ι		$(\iota\ \iota)$	ι
			ι			ι	
		$\iota_{\tau\omega}$			$\iota_{\tau\omega}$		
	$(o\iota\iota)$		ι			ι	
			o				
$o_{\tau\omega}$							

(S11) Charles has some brothers.

a) <u>Type-theoretical analysis</u>

Meaningful components: *Charles* - Ch, *brother* - B*, some* - $\exists^{\iota}$

Types of the denoted objects: Ch / ι, B / $(o\ \iota\ \iota)_{\tau\omega}$, $\exists^{\iota}$ / $(o\ (o\ \iota))$

We consider here the standard way of defining the semantics of *some,* namely as the expression denoting the existential quantifier, i.e. the class of non-empty classes (here: of individuals).

b) <u>Synthesis</u>

We first construct the property of being Charles's brother - CB / $(o\ \iota)_{\tau\omega}$ by:

$\lambda w \lambda t\ \lambda x\ [^0B_{wt}\ x\ ^0Ch]$ $\quad x \ldots \iota$

Now applying this property CB to a world W and time T we obtain the class of individuals who are Charles's brothers (in this W, T), and the sentence claims that this class is not empty.

The resulting construction (after applying the β-rule) is:

$$\lambda w \lambda t\ [^0\exists^{\iota}\ \lambda x\ [^0B_{wt}\ x\ ^0Ch]]$$

c) <u>Type-theoretical checking</u>

$$
\begin{array}{lllllll}
\lambda w \lambda t & [^0\exists^{\iota} & \lambda x & [^0B_{wt} & x & ^0Ch]] \\
 & & & (o\iota\iota) & \iota & \iota \\
 & & & & o & \\
 & (o(o\iota)) & (o\iota) & & & \\
 & o & & & & \\
o_{\tau\omega} & & & & &
\end{array}
$$

(S13) All Charles's friends are intelligent.

a) Type-theoretical analysis

Meaningful components: *all* - $\forall^{\iota}$, *Charles* - Ch, *friend* - F*, intelligent* - IN

Types of the denoted objects: $\forall^{\iota} / (o\ (o\ \iota))$, Ch / ι, F / $(o\ \iota\ \iota)_{\tau\omega}$, IN / $(o\ \iota)_{\tau\omega}$

We consider here the standard way of defining the semantics of *all,* namely as the expression denoting the general quantifier, i.e. the class of classes of all the members of a given type (here: of individuals - ι).

b) Synthesis

We can reformulate the sentence as follows: *It holds for all the individuals that if this individual is Charles's friend then this individual is intelligent.* This paraphrasing may seem to be rather peculiar, for there are no „if" and „then" in the original sentence, but if we want to interpret the expression *all* as the standard general quantifier, we have to do it in this way. There is, however, another possibility to analyse *all,* namely as denoting $\forall'$ / $(o(o\iota)(o\iota))$ - a function which when applied to a class C returns the class of all the classes which contain C as their subclass. Using the former variant, we *v*-construct a class

$\lambda x\ [\ {}^0\!\supset [{}^0F_{wt}\ x\ {}^0Ch]\ [{}^0IN_{wt}\ x]$

and apply the general quantifier $\forall^{\iota}$:

$[\forall^{\iota}\ \lambda x\ [\ {}^0\!\supset [{}^0F_{wt}\ x\ {}^0Ch]\ [{}^0IN_{wt}\ x]\ ]\ ]$

But the sentence is empirical, the above fact holds dependently on possible worlds and times, hence we have to make a closure using *w, t,* and the resulting construction is:

$$\lambda w \lambda t\ [\forall^{\iota}\ \lambda x\ [\ {}^0\!\supset [{}^0F_{wt}\ x\ {}^0Ch]\ [{}^0IN_{wt}\ x]\ ]\ ]$$

c) Type-theoretical checking

$$\begin{array}{cccccccccc}
\lambda w\lambda t & [\forall^{\iota} & \lambda x & [\ {}^0\!\supset & [{}^0F_{wt} & x & {}^0Ch] & [{}^0IN_{wt} & x]\ ]\ ] \\
 & & & & (o\iota\iota) & \iota & \iota & (o\iota) & \iota \\
 & & & (ooo) & & o & & & o \\
 & & & & & o & & & \\
 & (o(o\iota)) & (o\iota) & & & & & & \\
 & o & & & & & & & \\
o_{\tau\omega} & & & & & & & &
\end{array}$$

Note: Analyse the above sentence using the other quantifier $\forall'$ (without using ${}^0\!\supset$).

Hint: The expression *all Charles's friends* will be analysed by $[{}^0\forall'\ \lambda x\ [{}^0F_{wt}\ x\ {}^0Ch]]$ (see also the solution of S21).

(S15) The king of France does not exist

a) Type-theoretical analysis

Meaningful components: *the king* - K, *France* - F, *does not* - ¬, *exist* - E

Types of the denoted objects: K / $(\iota\,\iota)_{\tau\omega}$, F / ι, ¬ / (οο), E / $(\omicron\ \iota_{\tau\omega})_{\tau\omega}$

The types of K, F and ¬ are obvious. As for E, existence is a property of *intensions* not of individuals. If it were a property of individuals, then all the questions on existence would be *a priori* answered positively, since (in our conception of the universe of discourse) all the individuals are in all the possible worlds the same - „exist“. The property of existence is possessed by an intension in a world W and time T if at least one object plays the role determined by the intension; if the intension is an individual office (as it is in our example), then to exist means that the office is occupied (in a given world and time) by an individual.

b) Synthesis

The king of France denotes an individual office - the role an individual can play. The sentence claims that this office is not occupied in a given world-time. To construct the office we have to combine ^{0}K, ^{0}F:

$\lambda w \lambda t\ [^0K_{wt}\ ^0F]$.

The resulting construction is:

$$\lambda w \lambda t\ [^0\neg\ [^0E_{wt}\ [\lambda w \lambda t\ [^0K_{wt}\ ^0F]]\ ]$$

Note: The sentence could be also analysed by using the existential quantifier (x ... ι)

$\lambda w \lambda t\ [^0\neg\ [^0\exists\ \lambda x\ [^0{=}\ x\ [\lambda w \lambda t\ [^0K_{wt}\ ^0F]]_{wt}\ ]]]$, for the property E can be constructed as follows:

$\lambda w \lambda t\ \lambda c\ [^0\exists\ \lambda x\ [^0{=}\ x\ c_{wt}\ ]]$ (c ... $\iota_{\tau\omega}$).

c) Type-theoretical checking

$$\lambda w \lambda t \quad [^0\neg \quad [^0E_{wt} \quad [\lambda w \lambda t \quad [^0K_{wt} \quad ^0F]]\ ]$$

$(\iota\,\iota)$ ι

ι

$(\omicron\iota_{\tau\omega})$ $\iota_{\tau\omega}$

(οο) ο

ο

$\omicron_{\tau\omega}$

(S16) The most famous composer is bald.

a) <u>Type-theoretical analysis</u>

Meaningful components: *the most famous* - MF, *composer* - C*, being bald* - B

Types of the denoted objects: MF / $(\iota\ (o\iota)_{\tau\omega})_{\tau\omega}$, C / $(o\ \iota)_{\tau\omega}$, B / $(o\ \iota)_{\tau\omega}$

(B, C are obviously properties of individuals. MF is an intension which dependently on worlds-times associates a property with at most one individual - the most famous one.)

b) <u>Synthesis</u>

The most famous composer denotes an individual office - the role an individual can play. The sentence claims that the holder of this office in a given world-time (if any) belongs to the class of individuals that have (in that world-time) the property of being bald. To construct the office we have to combine ^{0}MF, ^{0}C:

$\lambda w \lambda t\ [{}^0MF_{wt}\ {}^0C]$.

Note that the property C is not applied here to *w, t*.

Applying this office to *w, t,* we obtain its holder: $[\lambda w \lambda t\ [{}^0MF_{wt}\ {}^0C]]_{wt}$.

The resulting construction is:

$$\lambda w \lambda t\ [{}^0B_{wt}\ [\lambda w \lambda t\ [{}^0MF_{wt}\ {}^0C]]_{wt}\ ]$$

or equivalently after the β-reduction:

$$\lambda w \lambda t\ [{}^0B_{wt}\ [{}^0MF_{wt}\ {}^0C]]$$

Note that the proposition constructed by this construction may not have any truth value in the actual world now, for the composition $[\lambda w \lambda t\ [{}^0MF_{wt}\ {}^0C]]_{wt}$ may be *v*-improper (if there were no holder of this office in the actual world now, for instance if there were two or more equally famous composers).

c) <u>Type-theoretical checking</u>

$$\lambda w \lambda t \quad [{}^0B_{wt} \quad [\lambda w \lambda t \quad [{}^0MF_{wt} \qquad {}^0C]]\ _{wt}\ ]$$

$(\iota\ (o\iota)_{\tau\omega}) \quad (o\iota)_{\tau\omega}$

ι

$\iota_{\tau\omega}$

$(o\iota) \qquad \iota$

o

$o_{\tau\omega}$

(S18) (All the) rich men are happy

a) Type-theoretical analysis

Meaningful components: *all* - $\forall^{\iota}$, *rich* - R, *man* - M*, happy* - H

Types of the denoted objects: $\forall^{\iota} / (o\ (o\ \iota))$, $R / ((o\ \iota)\ (o\ \iota)_{\tau\omega})_{\tau\omega}$, $M / (o\ \iota)_{\tau\omega}$

We consider here the standard way of defining the semantics of *all,* namely as the expression denoting the general quantifier, i.e. the class of classes of all the members of a given type (here: of individuals - ι). As for the type of R, it is an intension that in every world-time associates a property with a class (see D2 where the type of empirical adjectives has been explicated).

b) Synthesis

We can reformulate the sentence as follows: *It holds for all the individuals that if this individual has the property of being rich man then this individual is happy.* This paraphrasing may seem to be rather strange, for there are no „if" and „then" in the original sentence, but if we want to interpret the expression *all* as denoting the standard general quantifier, we have to do it in this way. There is, however, another possibility of analysing *all*, see example S13.

To construct the property of being rich man, we have to combine ${}^0R$ and ${}^0M$ as follows

$\lambda w \lambda t\ [{}^0R_{wt}\ {}^0M]$,

and to construct the class if individuals for whom the above condition holds (in a world-time) we have

$\lambda x\ [\ {}^0\supset [[\lambda w \lambda t\ [{}^0R_{wt}\ {}^0M]]_{wt}\ x]\ [{}^0H_{wt}\ x]$, or equivalently after the β-reduction

$\lambda x\ [\ {}^0\supset\ [[{}^0R_{wt}\ {}^0M]\ x]\ [{}^0H_{wt}\ x]\ ]$.

Now we apply the general quantifier $\forall^{\iota}$:

$[\forall^{\iota}\ \ \lambda x\ [\ {}^0\supset [[{}^0R_{wt}\ {}^0M]\ x]\ [{}^0H_{wt}\ x]\ ]$

But the sentence is empirical, the above fact holds dependently on possible worlds and times, hence we have to make a closure using *w, t,* and the resulting construction is:

$$\lambda w \lambda t\ [\forall^{\iota}\ \ \lambda x\ [\ {}^0\supset [[{}^0R_{wt}\ {}^0M]\ x]\ [{}^0H_{wt}\ x]\ ]\ ]$$

c) Type-theoretical checking

$$\lambda w \lambda t \quad [\forall^{\iota} \quad \lambda x \quad [\ {}^0\supset \quad [[{}^0R_{wt} \quad {}^0M] \quad x] \quad [{}^0H_{wt} \quad x]\ ]\ ]$$

$$((o\iota)(o\iota)_{\tau\omega}) \quad (o\iota)_{\tau\omega}$$

$$(o\iota) \quad \iota \quad (o\iota) \quad o$$

$$(ooo) \quad o \quad o$$

$$o$$

$$(o(o\iota)) \quad (o\iota)$$

$$o$$

$$o_{\tau\omega}$$

(S20) Honesty is desirable.

a) Type-theoretical analysis

Meaningful components: *honesty* - H, *desirable* - D

Types of the denoted objects: H / $(o\ \iota)_{\tau\omega}$, D / $(o\ (o\ \iota)_{\tau\omega}\ )_{\tau\omega}$

(H is a property of individuals. D is an intension ('of the second order'), it is a property of properties.)

b) Synthesis

The sentence claims that the property of being honest is desirable. Hence we have to apply the property D to world-time and afterwards to the property H.

The resulting construction is:

$$\lambda w \lambda t\ [{}^0D_{wt}\ {}^0H]$$

Note that the constituent ${}^0H$ occurs in this construction in the *de dicto* supposition, unlike the constituent ${}^0D$ which is in the *de re* supposition (for details see section F1).

c) Type-theoretical checking

$$\lambda w \lambda t \quad [{}^0D_{wt} \qquad {}^0H]$$

$$(o\ (o\iota)_{\tau\omega}) \qquad (o\iota)_{\tau\omega}$$

$$o$$

$$o_{\tau\omega}$$

(S21) Some politicians are honest

a) Type-theoretical analysis

Meaningful components: *some* - ∃, *politician* - P, *honest* - H.

H, P are properties of individuals. Analysing *some* as an expression denoting the standard existential quantifier - an (ο (οι))-object, has some unpleasant consequences: We have to reformulate the sentence as *There are some individuals who are politicians and who are honest.* Thus we have another object denoted by *and* - ∧ / (οοο). There is a more natural way of analysing *some*, namely as denoting ∃' / ((ο(οι)(οι)) - a function that when applied to a class C returns the class of all such classes which overlap with C.

Types of the denoted objects:

Variant 1: ∃ / (ο (οι)), P / $(ο\ ι)_{τω}$, H / $(ο\ ι)_{τω}$, ∧ / (οοο)

Variant 2: ∃' / ((ο(οι)(οι)), P / $(ο\ ι)_{τω}$, H / $(ο\ ι)_{τω}$.

b) Synthesis

Variant 1: The expressed construction is:

$$\lambda w \lambda t\ [{}^0\exists\ \lambda x\ [{}^0\wedge\ [{}^0P_{wt}\ x]\ [{}^0H_{wt}\ x]]]$$

Variant 2: The expressed construction is:

$$\lambda w \lambda t\ [[{}^0\exists'\ {}^0P_{wt}]\ {}^0H_{wt}]$$

Here the subconstruction $[{}^0\exists'\ {}^0P_{wt}]$ corresponds to the phrase *some politicians.* Note that the constituent ^{0}H occurs in both these constructions in the *de re* supposition (for details see section F1).

c) Type-theoretical checking

$\lambda w \lambda t \quad [{}^0\exists \quad \lambda x \quad [{}^0\wedge \quad [{}^0P_{wt} \quad x] \quad [{}^0H_{wt} \quad x]]]$

(οι) ι (οι) ι

(οοο) ο ο

ο

(ο(οι) (οι)

ο

$ο_{τω}$

$\lambda w \lambda t \quad [[{}^0\exists' \quad {}^0P_{wt}] \quad {}^0H_{wt}]$

(ο(οι)(οι)) (οι)

(ο(οι)) (οι)

ο

$ο_{τω}$

F. Applications (Some puzzling problems)

In this concluding section we use the TIL theory of constructions to solve some interesting problems connected with the logical analysis of natural language expressions. To make the text smooth and the constructions easier to read, we use from now on the following notational conventions and abbreviations:

Quantifiers Π^{α} - general and Σ^{α} - existential are (see Section B2) functional objects of a type ($o(o\alpha)$), singulariser I^{α} is an object of a type ($\alpha(o\alpha)$), for some type α. Instead of $[^0\Pi^{\alpha}\ \lambda x\ ...]$, $[^0\Sigma^{\alpha}\ \lambda x\ ...]$ we shall use the traditional notation $\forall x$ ..., $\exists x$ ... . Similarly instead of $[^0 I^{\alpha}\ \lambda x\ ...]$ we shall write ιx ... (the only x such that ...; we shall use the same sign 'ι' for the type of individuals as well as for singularisers, since no confusion arises). We shall also use the classical infix notation without trivialisation when writing logical connectives.

1. *De re / de dicto*

Problems connected with the analysis of using definite descriptions, i.e. expressions like "the president of USA", "the husband of Hillary Clinton", "the (present) king of France", etc., in the *de dicto / de re* supposition have been dealt with by many philosophers and logicians. A standard answer to the questions posed by this problem is an assertion that the meaning of such an expression depends on the context in which it is used.

Let us compare the following sentences:

(1) The president of USA is a member of Democratic party.

(2) The president of USA is eligible.

(3) Charles is seeking the president of USA.

(4) Charles believes that the president of USA is a member of Democratic party.

(5) The president of USA is such that Charles believes him to be a member of Democratic Party.

In all these sentences expression "the president of USA" is used, i.e. a definite description which according to traditional semantic theories (beginning with Frege) denotes a definite individual. But whereas in sentences (1) and (5) this expression can be substituted for by "Bill Clinton" without changing the truth-value of the sentence, in cases (2), (3), (4) it is impossible to do so correctly. Or, more precisely, whereas from the sentence (1) and from

(6) Bill Clinton is the president of USA.

it follows that Bill Clinton is a member of Democratic party, and from (5) and (6) it follows that Bill Clinton is such that Charles believes him to be a member of Democratic party, analogous inferences are not valid in cases (2), (3), (4). Moreover, there is an existential commitment connected with sentences (1) and (5). The existence of somebody who is the president of USA follows from these sentences, or (which is a stronger claim) the sentence

(7) The president of USA exists

is a presupposition of sentences (1) and (5), which again cannot be claimed about (2), (3), (4). One might be perhaps in doubt about the sentence (3). After all, when Charles is seeking the president of USA then he is looking for Bill Clinton, isn't he? (Providing Bill Clinton is the president, of course.) This is caused by a certain ambiguity of the homonymous expression "seeking" [Jespersen 1999]. If it is understood in the sense "wanting to locate in space and time", then he is really looking for Bill Clinton. We will, however, use it in the sense "wanting to find the holder of this office", similarly as in sentences like "Charles is seeking the murderer …", "Charles is seeking the unicorn", etc., and then everything that has been said above holds.

Hence what is the difference between using the expression "president of USA" in sentences (1), (5) and its using in sentences (2), (3), (4)? In contexts like (1) and (5) the truth-value of these sentences depends on the actual occupation of the office of the president of USA (*de re*) whereas in contexts like (2), (3), (4) it is not the case (*de dicto*). Adherents of traditional semantic theories therefore claim that the meaning of the expression "the president of USA" is context dependent. The "Fregeans" would probably say that in sentences (1) and (5) it denotes Bill Clinton whereas in oblique contexts like (3) and (4) it denotes its 'sense', Russell even deprives a definite description of its self-contained meaning. We have seen that the TIL approach is *anti-contextualistic*: such an expression denotes in all the contexts one and the same "thing", an individual office – a $\iota_{\tau\omega}$-object, never its holder in the actual possible world now, and has a definite meaning which is best explicated by the respective construction.

The Fregean sense/reference semantics that still in a way dominates the field fails to distinguish supposition from reference. Using the term "the president of USA" in the former context differs from the latter in the *de re* / *de dicto* supposition in which it occurs. The term is in no way ambiguous. The respective analysis of the above sentences reveals this fact. We first perform the simplest case, namely the analysis of (1) and (2), for analysing (3), (4), (5) will first require a short reflection on the problem of propositional / notional attitudes. Let us assume for the sake of simplicity that the expression „the president of USA" is connected with a simple concept ^0P where P is the individual office, a $\iota_{\tau\omega}$ -object. D(emocrat) is a property of individuals, an $(o\iota)_{\tau\omega}$-object, whereas E(ligible) is a property of individual offices, an $(o\ \iota_{\tau\omega})_{\tau\omega}$ - object. Sentence (1) now expresses the following construction (w ... ω, t ... τ)

(1') $\lambda w \lambda t\, [{}^0D_{wt}\ {}^0P_{wt}]$

whereas sentence (2) represents

(2') $\lambda w \lambda t\, [{}^0E_{wt}\ {}^0P]$.

We can see that in (1') the office P is applied to a world *w* and time *t*, there is an intensional descent here, and the sentence claims that the holder of the office of the president of USA (whoever it may be) has the property of being a member of Democratic Party (DP). The sentence speaks about the office P but the property of being a member of DP cannot be ascribed to the office but to an individual. The office P plays here the role of a „pointer" to an unspecified „thing" (*res*) that is a member of DP. (Needless to say that the sentence does not say anything about Bill Clinton.) The truth-value of the sentence depends on which individual holds the office P in the actual world and time. We will say that the constituent ${}^0P$ is in (1') in the *de re* supposition. (Similarly we can say that the property D plays the role of a pointer to an unspecified „thing" (*res*) - class of individuals and the constituent ${}^0D$ is also in (1') in the *de re* supposition.)

On the other hand, the sentence (2) speaks about the office P itself (*dictum*), for in (2') there is no intensional descent of P. The truth-value of the sentence does not depend on which individual (if any) holds the office P in the actual world and time. We will say that the constituent ${}^0P$ is in (2') in the *de dicto* supposition.

Note: To be precise, in (2) we should not speak about the president of USA but about the USA *presidency*. But (2) is not only admissible but also quite common, whereas the sentence 'The USA presidency is eligible' is rather spurious.

Now the claims about the existential commitment and presupposition in the *de re* case are obvious. In those worlds *w* and times *t* where the construction ${}^0P_{wt}$ is improper (there is no president), the proposition constructed by (1') does not have any truth-value. If the sentence (1) has a truth-value (True or False), then the president of USA exists. It is also easy to show that in the *de re* case we can substitute another definite description for the expression 'the president of USA', namely the description with the same reference, without changing the truth-value of the sentence. Thus, e.g., the following inference is valid

(1) The president of USA is a member of Democratic party

(8) The president of USA is the husband of Hillary Clinton

hence

(9) The husband of Hillary Clinton is a member of Democratic party,

for (Hu(sband) / $(\iota\ \iota)_{\tau\omega}$, H(illary) / ι, the office 'the husband of Hillary ...' is constructed by means of $\lambda w \lambda t\ [^0Hu_{wt}\ ^0H]$)

(1') $\lambda w \lambda t\ [^0D_{wt}\ ^0P_{wt}]$

(8') $\lambda w \lambda t\ [^0{=}\ ^0P_{wt}\ [\lambda w \lambda t\ [^0Hu_{wt}\ ^0H]_{wt}\ ]$

(9') $\lambda w \lambda t\ [^0D_{wt}\ [\lambda w \lambda t\ [^0Hu_{wt}\ ^0H]_{wt}\ ]$

In all the worlds and times in which propositions constructed by (1') and (8') are true the proposition constructed by (9') is true as well.

Now we can generalise our considerations and define:

Definition 13. (the *de re / de dicto* supposition)

A subconstruction C of a construction C' is in the *de re supposition* in the construction C' iff the value of the intension constructed by C' in a world W and time T depends on the value of the intension constructed by C in this W, T, otherwise C is in C' in the *de dicto supposition.*

We then also speak about the (*de re / de dicto) supposition* of the expression E that represents the construction C.

Note: Note that in accordance with this definition the constructions (1') and (2') are in themselves in the *de re* supposition.

Definition 13 could still be generalised as follows: A construction C is in the *de re* supposition in a construction C', if the value of the function constructed by C' depends on the value of the function F constructed by C, otherwise (if it depends on the whole function F) C is in C' in the *de dicto* supposition.

Example:

$\lambda x\ [^0{>}\ [^0\cos x]\ ^00]$ (*de re*)

$[^0\text{Periodic}\ ^0\cos]$ (*de dicto*)

In the following text we will, however, deal only with empirical expressions that denote intensions and use the definition 13.

2. Propositional / notional attitudes

The problem of attitude reports has been thoroughly discussed by many philosophers and logicians. Though it is a familiar ground to almost everybody who is involved in providing a formal account of the semantics of a natural language, let us briefly recapitulate why there exists a problem at all regarding the semantics of propositional and notional attitude reports. The goal of formal semantics is to assign meanings systematically to (reasonable) expressions of a natural language. What meanings are taken to be varies from theory to theory but some version of the principle of compositionality (the meaning of a composed expression is a function of the meanings of its parts) should be always taken into account. But it is just the attitude reports which seemingly appear as if not obeying the desirable principle of compositionality. If we were content to give a semantic account of a sufficiently restricted subset of a natural language, excluding modalities and propositional attitudes, among others, it might seem to be sufficient to conceive meanings of sentences to be simply their truth-values. But as soon as we take into account modal or attitude contexts, substitution of embedded sentences with the same truth-value does not preserve the truth-value of the embedding sentence. The problem was noticed already by G. Frege in his article [Frege 1892] where he had to admit that his classical denotational approach was untenable. (There is obviously something deeply counterintuitive about the claim that the meaning of a sentence is its truth-value but it was just the case of „indirect contexts" which made Frege to be aware of the general untenability of this claim.) His well-known solution consisting in splitting the meaning into sense and reference can be called *contextual approach* and it suffers from two flaws, namely its contextualism and its not specifying what the sense is.

The standard response to this problem has been to take the meaning of a sentence to be the set of possible worlds in which it is true (or, more precisely, a proposition denoted by the sentence, i.e. a partial function mapping the set of possible worlds into a chronology of members of the type {True, False}). The use of such approach originated with Montague [Montague 1974] and has been a dominant approach in the field. Possible-world semantics yields more plausible meanings of sentences than the standard 'Fregean-like' semantics, and it is able to solve the problem of modal contexts. (In the cases like „It is necessary that S", where S denotes the trivial constant proposition TRUE - true in all possible worlds and time points, the substitution of any empirical sentence S' for S is blocked, for S' denotes a non-trivial proposition.) Yet propositional-attitude reports require an even narrower individuation of meanings of sentences in order to pass the substitution test. This is extremely clear in the case of mathematical sentences, where this approach links every (true) mathematical sentence with the constant proposition TRUE, and the well-known paradox of omniscience is unavoidable. (David Hilbert, e.g., as a competent mathematician, certainly believed many mathematical truths, without believing that the first-order logic is undecidable.)

It might seem that the above example concerns only the case of an analytic embedded sentence, i.e. a sentence denoting a constant (true or false) proposition, and that in the case of empirical embedded sentences what is believed, known, etc. is the state-of-affairs referred to by the embedded sentence, i.e. a proposition. This would lead to a dualism: the type of propositional attitudes to mathematical (analytical) clauses would differ from the type of attitudes to empirical clauses. This dualism was defended in [Materna 1984] and rejected in [Duží 1996] and [Materna 1998] for the following reason: There are (theoretically) infinitely many equivalent transformations of one and the same sentence and, indeed, „X Bs that S“ and „X Bs that S’“ (where B is an attitude verb such as believe, know, realise, and S, S’ are embedded sentences) can differ in truth-value even if S and S’ are logically equivalent (i.e. denote one and the same proposition). A student can easily believe that it is not true that if A then B without believing that A and not B. It is not at all a problem of linguistic understanding that students are unable to draw some inferences. However, as we show below, a certain kind of such a dualism can be defended in the case of notional attitudes, and, in a way also in the case of propositional attitudes (see also [Jespersen 1998]).

Now a question arises: How fine-grained must meanings be? The obvious structural sensitivity of the attitude reports suggests that meanings should be individuated almost as finely as sentences of a natural language to avoid undesirable entailments among attitude reports. Yet we do not accept theories of the attitudes as relations to syntactic objects, the simplest form of which is sententialism ([Quine 1990], in a way [Stalnaker 1984]). A thorough criticism of Quine’s approach can be found in [Tichý 1988, p.12-14], showing its counterintuitivity; there is also a problem of quantifying into attitude contexts that is particularly troublesome for syntactic theories ([Moore 1995]).

Carnap [Carnap 1947] was also aware of the problem of attitude reports. He says that belief sentences are neither extensional nor intensional with respect to the subsentence S, and the explication of these contexts requires a stronger meaning relation than identity of intension. (In [Bäuerle, Cresswell 1989] these contexts are referred to as *hyperintensional* contexts.) According to Carnap, two sentences S and S’, to be mutually substitutable in belief sentences, must have more in common than intension, they must have the same *intensional structure*. His relation of intensional isomorphism is more plausible for giving an interpretation of belief sentences avoiding the so-called paradox of analysis than, e.g., Quine’s approach. Yet it has been criticised already by A. Church in [Church 1954], where two predicate constants are introduced which are L-equivalent but not synonymous, and all the same creating two intensionally isomorphic sentences - one of them certainly believed to be true while the other not. Church proposes that ‘synonymous isomorphism’ should replace Carnap’s intensional isomorphism as a criterion of identity of belief. We cannot but agree, still, as he admits, it remains being necessary to provide a determination of synonymy (for such a definition see [Duží 1996]). Another example of a general insufficiency of Carnap's solution can be found in [Tichý 1988, pp.8-9].

There are many other attempts to solve the problem of attitudes, to name at least Cresswell's 'structured meanings' [Cresswell 1985] or Zalta's 'abstract objects' [Zalta 1988], the criticism of the former can be found in [Tichý 1994] and of the latter in [Materna 1998], and it is out of the scope of this study to give an exhaustive and critical list of them. A good survey can be found, e.g., in [Bäuerle, Cresswell 1989] and an outline of the history of these attempts see, e.g., [Aho 1994]. To sum up, we look for structured entities that are encoded by linguistic expressions, that are objective, independent of a particular language, and that serve as an explication of Frege's sense, i.e., that stand for the meaning of an expression. In our opinion, as we have shown in the previous sections, the most adequate solution overcoming the flaws of the above mentioned approaches is offered by Tichý's theory of *constructions*.

2.1. Propositional attitudes

Using constructional approach, we can easily see that there is no paradox in student's believing that it is not true that if A then B without his believing that A and not B. The student can master language perfectly well but he lacks knowledge of an elementary logical law; we have to teach him that the two embedded that-clauses are equivalent. Since they express different constructions, they do not have the same meaning and are, therefore, not intersubstitutable. This seems to be indisputable. Yet a question can arise whether this constructional approach is not in a way too restrictive, too fine-grained. M. Richard in [Richard 1990, p.17,18] says: „In general, if it is just *possible* that X believes that A be true while X believes that B not be true, then we have to assign the terms 'that A' and 'that B' different things". But at the same place he claims that „two 'that-clauses' name distinct things if they have distinct *enough* structure: The structures of sentences of the forms 'A and B', 'B and A' *may* not be different enough while the structures of 'not (A ⊃ B)', 'A and not B' are." Applying our conception consequently, we get that 'A and B', 'B and A' express two *different* constructions ($^0[A \wedge B] \neq {}^0[B \wedge A]$), hence we have to admit that it is possible to believe that A and B while not believing that B and A.

Another context in which constructional approach might seem to be too restrictive is the problem of existential quantification into 'intentional contexts'. Our logical intuition says that (2) logically follows from (1):

(1) Charles believes that 7537 is a prime number.

(2) There is a number x such that Charles believes that it (x) is a prime number.

But it can be easily shown that in this case constructional approach is not too restrictive, just opposite, it is fully justified and necessary; it reveals the core of non-triviality of the problem of existential quantification into intentional contexts (EQI): The analysis of (1) is as follows:

(1') $\lambda w \lambda t\, [{}^0B_{wt}\, {}^0Ch\, {}^0[{}^0Pr\, {}^0 7537]]$ (where B(elieve)/$(o\iota *_1)_{\tau\omega}$, Ch(arles)/$\iota$, Pr(ime)/$o\tau$)

Now trying to analyse (2), we get into troubles. The first attempt might look like (2'):

(2') $\lambda w \lambda t\, \exists x\, [{}^0B_{wt}\, {}^0Ch\, {}^0[{}^0Pr\, x]]$ (x ranging over τ)

which is certainly not correct. Variable x occurring „inside“ the context is here o-bound by the outer trivialisation and cannot be therefore quantified. Indeed, this is quite comprehensive and correct: the variable is here *mentioned* rather than used. (If the constructional approach were not used, the analysis of (2) would be seemingly easier; we would carelessly „quantify into“ the context, which would not be correct just for the above reason.) The way out consists in using two special functions, namely substitution function Sub^n and trivialisation function Tr^α (see Section D4):

Let C, d, E be constructions of order n, d be a variable. The function Sub^n, type $(*_n *_n *_n *_n)$, is a mapping which, applied to (C, d, E), returns the construction which is the result of (correctly) substituting C for d in E.

Let α be a type of order n, a be an object of the type α; the function Tr^α, type $(*_n\, \alpha)$, applied to a, returns the trivialisation of a. (So that, e.g., $[{}^0Tr^\tau\, {}^0 2]$ constructs ${}^0 2$, $[{}^0Tr^\tau\, x]$ v-constructs the trivialisation of the number v-constructed by x.)

Now the correct analysis of (2) is as follows:

(2'') $\lambda w \lambda t\, \exists x\, [{}^0B_{wt}\, {}^0Ch\, [{}^0Sub^1\, [{}^0Tr^\tau\, x]\, {}^0x\, {}^0[{}^0Pr\, x]]]$

The entailment of (2'') from (1') is justified: Let W, T be a possible world and time point, respectively, such that for the valuation v: $w \rightarrow W$, $t \rightarrow T$ the construction $[{}^0B_{wt}\, {}^0Ch\, {}^0[{}^0Pr\, {}^0 7537]]$ v-constructs True. Then there is an object of the type τ the trivialisation of which can be substituted for x in $[{}^0Pr\, x]$, viz. the number 7537.

Consider another inference - the well-known Quine's example [Quine 1956]: From

(3) Ralph believes that the tallest spy is a spy

we should be able to infer

(4) Something is such that Ralph believes it to be a spy.

This example has been the subject of much dispute [Zalta 1988]. The problem is what to say about the situation in which Ralph does not have a slightest idea who the tallest spy is and the inference in question seems to fail. Of course, his belief cannot concern a definite individual. It concerns instead an individual office, an $\iota_{\tau\omega}$-object, constructed by $[\lambda w\lambda t\ [^0Tal_{wt}\ ^0S_{wt}]]$ (S - an $(o\iota)_{\tau\omega}$-object, a property of being a spy, Tal - a $(\iota(o\iota))_{\tau\omega}$-object, an intension which associates with every world-time couple the function that selects the tallest individual (if any) from any class of individuals). Now there is no failure of inference here. From the proposition constructed by

(3') $\lambda w\lambda t\ [^0B_{wt}\ ^0R\ ^0[\lambda w\lambda t\ [^0S_{wt}\ [\lambda w\lambda t\ [^0Tal_{wt}\ ^0S_{wt}]]_{wt}]]]$

the proposition constructed by (4') follows:

(4') $\lambda w\lambda t\ \exists c\ [^0B_{wt}\ ^0R\ [^0Sub^1\ [^0Tr\ c]\ ^0c\ ^0[\lambda w\lambda t\ [^0S_{wt}\ c_{wt}]]]]$

(where variable c ranges over $\iota_{\tau\omega}$).

The next example deals with the problem of believing in something that does not exist. Existence has a special feature, namely the fact that if existence were ascribed to individuals, then all the existential questions would be *a priori* answered positively. Since any individual is trivially one of the individuals that there are, all the ascribing of the attribute of existence would be tautologically true. Some philosophers who deal with this peculiarity of existence usually come to the conclusion that it is simply not a property belonging to things (Kant, Russell, Aristotle). In TIL's conception, existence is quite a proper property of things, but not a property acquired by individuals but by offices that can be occupied by individuals [Tichý 1979]. Consider Charles's belief in (5):

(5) Charles believes that Santa Claus exists.

From (5) we would like to infer (6):

(6) There is something such that Charles believes it to be existing.

To prove it we again cannot analyse „something" as an individual. Charles's belief does not concern an individual, which one could it be? The subject of his belief is an *office* of Santa Claus (SCl), an $\iota_{\tau\omega}$-object, and he believes that this intension is occupied in the actual possible world and time. So the existence (E) is in this case a property of an individual office: an object of the type $(o\iota_{\tau\omega})_{\tau\omega}$ and proper analysis of (5) comes as follows:

(5') $\lambda w\lambda t\ [^0B_{wt}\ ^0Ch\ ^0[\lambda w\lambda t\ [^0E_{wt}\ ^0SCl\ ]]]$,

from which (6') can be inferred (c ranging over $\iota_{\tau\omega}$):

(6') $\lambda w\lambda t\ \exists c\ [^0B_{wt}\ ^0Ch\ [^0Sub^1\ [^0Tr\ c]\ ^0c\ ^0[\lambda w\lambda t\ [^0E_{wt}\ c]]]]$

This can be read as „There is an individual office c such that Charles believes that c is occupied in the actual world-time".

The (non-) existence can be ascribed not only to (individual) offices but also to properties (Golden mountains don't exist) and the situation is quite analogous.

But what about the sentence „The greatest number doesn't exist?" Now we don't ascribe non-existence to an office of the greatest number (it would be a „degenerated" office, $\tau_{\tau\omega}$-object, that would not be occupied in any world-time). We simply claim that the concept of the greatest number is strictly empty, it does not identify any object ([Duží, Materna 1994]). The respective construction $\iota x\ (\forall y\ [^0\!\geq xy])$ does not construct anything, it is improper. Yet from

(7) Charles believes that there is the greatest number

we can infer that

(8) There is something such that Charles believes it to exist.

Now Charles's belief doesn't concern even an office; the subject of his attitude is the respective *concept* of the greatest number represented by the above construction. Since variables ranging over constructions of the given order are at our disposal (due to the ramified hierarchy of types), we can construct an existential generalisation over one of such variables. Analysing (7) as

(7') $\lambda w\lambda t\ [^0B_{wt}\ ^0Ch\ ^0[\exists z\ [^0\!=z\ \iota x(\forall y[^0\!\geq xy])]]$

we can trivially infer

(8') $\lambda w\lambda t\ \exists c\ [^0B_{wt}\ ^0Ch\ \ c]$

where $c/*_2$ is a variable ranging over $*_1$, x,y,z range over τ.

Yet a deeper analysis is possible. Let E be the class of improper constructions of order 1, that is an $(o*_1)$-object. Then (7) can be understood as a claim about the construction $\iota x(\forall y[^0\!\geq xy])$:

(7'') $\lambda w\lambda t\ [^0B_{wt}\ ^0Ch\ ^0[\ \neg[^0E\ ^0[\iota x(\forall y[^0\!\geq xy])]]]]$

and the entailed construction will be

(8'') $\lambda w\lambda t\ \exists c\ [^0B_{wt}\ ^0Ch\ [^0Sub^2\ [^0Tr\ c]\ ^0c\ ^0[\ \neg\ [^0E\ c]]]]$

Now we can raise the question concerning a certain kind of dual approach to propositional attitudes. Summarising what has been said above, if the dependent clause is a non-empirical claim (for example an arithmetical sentence), there is simply no other possibility than to represent this claim by a construction - otherwise the paradoxes like that of omniscience are inevitable. But as soon as the dependent clause is an empirical sentence the situation is not that simple. The argument that also in this case the attitude relates an individual with a construction is supported by the fact that a sufficiently complex L-transformation of the dependent clause A leads to a very complicated construction, so that the believer, knower, etc. may change his/her attitude when the same proposition is denoted by the L-equivalent sentence B. Yet this 'empirical case' differs from the 'non-empirical case' in one respect: we can interpret such an attitude as the attitude to the 'state of the world', independently of the way the respective proposition has been constructed. So the believer etc. can possess this attitude without knowing that he/she does. If somebody believes that Mt Everest is higher/lower than Mt Blanc, then his/her attitude concerns the proposition itself, and if this proposition is represented by a very complicated equivalent sentence (expressing, therefore, a much more complicated construction), then dissenting with this sentence cannot influence his/her conviction that ... . So we can conceive attitude expressions as systematically ambiguous, denoting either $(o\ \iota\ *_n)_{\tau\omega}$ -, or $(o\ \iota\ o_{\tau\omega})_{\tau\omega}$ - objects. In what follows (namely when analysing the *de re* attitudes) we will use the latter possibility.

2.2. Notional attitudes

Notional attitudes can be also analysed by means of constructional approach. However, in this case a certain kind of dualism can be conceded. Attitudes to mathematical notions are indubitably attitudes to constructions, whereas attitudes to empirical notions can be considered to be explicated in at least three ways: as a relation (-in-intension) to a construction, or to an intension, or even to an individual.

As an example of the former, we would like to be able to prove that (10) follows from

(9) Charles calculates 2 + 5.

(10) Charles calculates something.

This is a simple case. Calculating (Calc) is an $(o\iota*_1)_{\tau\omega}$-object and having a variable *c* ($c/*_2$, *c* ranging over $*_1$), we get

(9') $\lambda w \lambda t\ [{}^0Calc_{wt}\ {}^0Ch\ {}^0[{}^0+\ {}^02\ {}^05]]$

(10') $\lambda w \lambda t\ \exists c\ [{}^0Calc_{wt}\ {}^0Ch\ c]$

Indeed, there is a valuation that assigns $[{}^0+\ {}^02\ {}^05]$ to *c* and the inference is obviously valid.

At the same time this constructional approach blocks undesirable invalid inferences like

Charles calculates 2 + 5.

2 + 5 = 7

Hence Charles calculates 7.

The substitution is blocked, since ${}^0[{}^0+\ {}^02\ {}^05] \neq {}^0[{}^07]$; the second premise claims only an equivalence of the two constructions ($[{}^0+\ {}^02\ {}^05] = {}^07$), not their identity.

Another simple example of an undeniably constructional attitude is

(11) Charles is thinking about the greatest number

It must be an attitude to a construction - $(o\iota*_1)_{\tau\omega}$ - object (what else could he be thinking about?), and

(12) There is something Charles is thinking about,

namely construction:

(11') $\lambda w \lambda t\ [{}^0\text{Think}_{wt}\ {}^0\text{Ch}\ {}^0[\iota x(\forall y[{}^0\geq xy])]]$

(12') $\lambda w \lambda t\ \exists c\ [{}^0\text{Think}_{wt}\ {}^0\text{Ch}\ c]$ (c ranging over $*_1$)

(We let aside the problem of polymorphic character of such attitude verbs like think, contemplate, etc. which has been dealt with in [Duží 1998].)

Applying the same approach to attitudes to empirical notions does not seem so straightforward any more. Let us exemplify such attitudes like seeking, hating, believing in, worshipping, and so like. We may conceive them again as constructional attitudes, $(o\ \iota\ *_1)_{\tau\omega}$-objects. Consider, e.g.,

(13) Charles is seeking the murderer of his father

and its respective analysis - S(eeking) / $(o\ \iota\ *_1)_{\tau\omega}$, Ch / ι, M(urderer) / $(o\ \iota\ \iota)_{\tau\omega}$, F(ather) / $(\iota\ \iota)_{\tau\omega}$

(13') $\lambda w \lambda t\ [{}^0\text{S}_{wt}\ {}^0\text{Ch}\ {}^0[\lambda w \lambda t\ \iota x\ [{}^0\text{M}_{wt}\ x\ [{}^0\text{F}_{wt}\ {}^0\text{Ch}]]]]$

Now we can infer - there is a construction to which Charles has an attitude of seeking. But isn't it rather weak and unnatural? Wouldn't we rather say that Charles's seeking concerns an individual office? We can certainly infer more:

(14) There is an individual office such that Charles wants to find its holder (in the actual world/time).

We get the logical consequence of (13'):

(14') $\lambda w \lambda t\ \exists c\ [{}^0\text{S}_{wt}\ {}^0\text{Ch}\ [{}^0\text{Sub}^1\ [{}^0\text{Tr}\ c]\ {}^0c\ {}^0c]]$ (c ranging over $\iota_{\tau\omega}$)

(Of course, we don't claim that there is an *individual* such that .... There is no existential commitment in this entailment.)

Now the question arises again. Isn't constructional approach too restrictive in this case? Consider a simple variant of (13):

(15) Charles is seeking the murderer of his father and his mother.

We will certainly claim that from (15) it follows that

(16) Charles is seeking the murderer of his mother and his father.

Yet this inference is blocked (Mo(ther) / $(\iota\ \iota)_{\tau\omega}$):

${}^0[\lambda w \lambda t\ \iota x\ ([{}^0\text{M}_{wt}\ x\ [{}^0\text{F}_{wt}\ {}^0\text{Ch}]] \wedge [{}^0\text{M}_{wt}\ x\ [{}^0\text{Mo}_{wt}\ {}^0\text{Ch}]])] \neq$

${}^0[\lambda w \lambda t\ \iota x\ ([{}^0\text{M}_{wt}\ x\ [\text{Mo}_{wt}\ {}^0\text{Ch}]] \wedge [{}^0\text{M}_{wt}\ x\ [{}^0\text{F}_{wt}\ {}^0\text{Ch}]])]$

Well, we might introduce an equivalence relation on the set of constructions, a stronger one than that induced by the identity of a constructed object (and still weaker than Materna's QUID relation, see [Materna 1988]), and claim that the conclusion can be deduced when the respective constructions are not „distinct too much“, i.e. when they are members of one and the same equivalence class. But there are no precise criteria for specifying such an equivalence relation.

Tichý characterises the relation between the seeker and what is being sought as a relation-in-intension between an individual and an individual office, i.e. as an $(\omicron\ \iota\ \iota_{\tau\omega})_{\tau\omega}$-object ([Tichý 1988, pp.214,215]). Conceiving seeking in this way still does not break the principle formulated by Jespersen in [Jespersen 1998]:

> The activity of seeking must be radically independent of questions of existence/ uniqueness and numerical identity. Otherwise a seeker would immediately be a finder as well.

Of course, the search could take place even if the office in question were vacant, if Charles's father were not actually murdered, or if Charles were seeking a unicorn. Now the analysis of (13) is as follows:

(13'') $\lambda w\lambda t\ [^0\mathrm{Seek}_{wt}\ ^0\mathrm{Ch}\ [\lambda w\lambda t\ \iota x\ [^0\mathrm{M}_{wt}\ x\ [^0\mathrm{F}_{wt}\ ^0\mathrm{Ch}]]]]$

The consequence (14) gets the analysis (14''):

(14'') $\lambda w\lambda t\ \exists c\ [^0\mathrm{Seek}_{wt}\ ^0\mathrm{Ch}\ c]$ (c ranging over $\iota_{\tau\omega}$).

Note that the construction of the office of the murderer is in the *de dicto* supposition. Seeking, thus conceived, is not a relation that an individual bears to another individual. In seeking the murderer, Charles certainly hopes to eventually find an individual, but before he succeeds he does not know which individual, if any, it will turn out to be. Even if he met him he would ignore him until he would discover the connection between the office and its particular holder. The above objection is now overcome: „the murderer of his father and his mother“ and „the murderer of his mother and his father“ denote one and the same office -

$[\lambda w\lambda t\ \iota x\ ([^0\mathrm{M}_{wt}\ x\ [^0\mathrm{F}_{wt}\ ^0\mathrm{Ch}]] \wedge [^0\mathrm{M}_{wt}\ x\ [^0\mathrm{Mo}_{wt}\ ^0\mathrm{Ch}]])] =$

$[\lambda w\lambda t\ \iota x\ ([^0\mathrm{M}_{wt}\ x\ [^0\mathrm{Mo}_{wt}\ ^0\mathrm{Ch}]] \wedge [^0\mathrm{M}_{wt}\ x\ [^0\mathrm{F}_{wt}\ ^0\mathrm{Ch}]])]$,

substitution is justified and the entailment is guaranteed.

Yet this rather „coarser“ solution is apparently blocking enough. From premises (13) and

(17) The murderer of Charles's father is our gardener

it does not follow that Charles is looking for the gardener; (17) expresses

(17') $\lambda w\lambda t\ [^0{=}\ [\lambda w\lambda t\ \iota x\ [^0\mathrm{M}_{wt}\ x\ [^0\mathrm{F}_{wt}\ ^0\mathrm{Ch}]]]_{wt}\ ^0\mathrm{G}_{wt}]$,

which states the contingent identity of the holders of two *different* offices, not of the offices themselves. The constructions of both offices occur here in the *de re* supposition. Hence the substitution is blocked.

Let us remark in this place that Russell's elimination of definite descriptions as expressions with a self-contained meaning [Russell 1959] is not correctly applicable, for it would lead to an undesirable existential commitment. Paraphrasing the sentence (13) according to Russell we would obtain:

There is an *x* such that *x* is the murderer of Charles's father and this *x* is the only one and Charles is looking for it.

This sentence is false in case there is no murderer, though the sentence (13) can be true.

On the other hand some other attitudes like, e.g., hating, loving, worshipping are obviously (similarly as to kick, to find, etc.) relations of an individual to an individual, i.e. $(o\ \iota\ \iota)_{\tau\omega}$ - objects. If our Charles hates the murderer of his father, we would be inclined to say that he bears a relation of hating to a particular individual though he may not know who the individual is, „via" the respective office, of course. He might reasonably say, e.g., whoever he is, I hate *him*. Similarly, when somebody worships the mayor of Dunedin, he does not seem to be only in a relation to an abstract function - office, but to a person, whoever it is, who copes the office of the mayor. Thus „hating", „worshipping" stand for an $(o\ \iota\ \iota)_{\tau\omega}$ - object. It is possible that the person in question hates, worships the office as such (the *de dicto* 'interpretation') but since in this case the *de re* 'interpretation' seems to be much more plausible, let us analyse the consequences.

We have to admit, that from premises (18) and (19) the conclusion (20) follows:

(18) Charles hates the murderer of his father

(19) The murderer of his father is our gardener

(20) Charles hates the gardener

For we get - H(ates) / $(o\ \iota\ \iota)_{\tau\omega}$:

(18') $\lambda w \lambda t\, [{}^0H_{wt}\ {}^0Ch\ [\lambda w \lambda t\ \iota x\ [{}^0M_{wt}\ x\ [{}^0F_{wt}\ {}^0Ch]]]_{wt}]$

or equivalently (after the β-reduction)

(18'') $\lambda w \lambda t\, [{}^0H_{wt}\ {}^0Ch\ \iota x\ [{}^0M_{wt}\ x\ [{}^0F_{wt}\ {}^0Ch]]]\ ]$

(19') $\lambda w \lambda t\, [{}^0{=}\ \iota x\ [{}^0M_{wt}\ x\ [{}^0F_{wt}\ {}^0Ch]]\ {}^0G_{wt}]$

(20') $\lambda w \lambda t\, [{}^0H_{wt}\ {}^0Ch\ {}^0G_{wt}]$

The construction of the office of the murderer in (18') is in *de re* supposition. The truth-value of the proposition constructed by (18') in the actual possible world now depends, of course, on the value of this office in the actual world now. The intersubstitutivity principle is valid, substitution is allowed on the basis of (19'). It is now nearly dispensable to remind the reader of the fact that in this case (the *de re* supposition) there is the existential commitment, the existence of the murderer of Charles's father is the presupposition of (18) and the inferences like the above, when another definite description with the same reference is substituted for 'the murderer of Charles's father' are valid. Thus from (18') it follows that

$\lambda w \lambda t\ \exists y\ [\ [^0\!=\ y\ \iota x\ [^0M_{wt}\ x\ [^0F_{wt}\ ^0Ch]]] \wedge [^0H_{wt}\ ^0Ch\ y]\ ]$

(There is somebody who is the murderer of Charles's father and Charles hates him, though he may not know it, we know it „from outside".)

Let us consider once again the *de dicto* case. If (13) and (17) hold, then we can obviously claim that

The gardener is such that Charles is seeking him,

by which we, of course, do not claim that Charles is seeking the gardener. Hence the above sentence follows from (13) and (17). How to give a reason now for such a valid inference in the *de dicto* case? There is probably no other solution but to paraphrase this sentence as follows:

The holder of the office of the gardener is the same individual as the holder of the office to which Charles is in the search relation,

the analysis of which reveals the fact of the inference (variable *c* ranging over $\iota_{\tau\omega}$):

$\lambda w \lambda t\ \exists c\ (\ [^0Seek_{wt}\ ^0Ch\ c] \wedge [^0\!=\ c_{wt}\ ^0G_{wt}]\ )$

3. *De re / de dicto* continued

Till now we have dealt with cases when definite descriptions (denoting offices) occurred in the *de dicto / de re* supposition. Analogous duality can be, however, observed in case of sentences or propositional constructions. Consider now another variant of sentences (1) and (4) from the above *de re / de dicto* section F1:

(1) The king of France is bald

(2) Charles thinks that the king of France is bald.

Let us perform the analysis, and for the sake of simplicity consider 'the king of France' as expressing a simple concept ^{0}KF, where KF is the individual office, $\iota_{\tau\omega}$ - object, B(ald) / $(o\iota)_{\tau\omega}$, Ch / ι. In the above section F2 on propositional attitudes we defend the solution which takes these attitudes as attitudes to *constructions*, thus for instance thinking is a relation-in-intension between an individual and a construction of proposition (not taking into account the polymorphic character of the expression 'think' [Duží 1995]), i.e. an object of type $(o\ \iota\ *_1)_{\tau\omega}$. But we have also shown that these attitudes can be conceived as attitudes to the states of affairs, i.e. to propositions, and here, for the sake of simplicity, we will consider this possibility, which will not influence the generality of our considerations about the *de re / de dicto* suppositions. Hence Th(inking) will be conceived as $(o\ \iota\ o_{\tau\omega})_{\tau\omega}$ - object.

(1') $\lambda w \lambda t\ [{}^0B_{wt}\ {}^0KF_{wt}]$

(2') $\lambda w \lambda t\ [{}^0Th_{wt}\ {}^0Ch\ [\lambda w \lambda t\ [{}^0B_{wt}\ {}^0KF_{wt}]]]$

The proposition P constructed by (1') does not have in the actual world now any truth-value, because ${}^0KF_{wt}$ is *v*-improper - the king of France does not exist now. The constituent ${}^0KF$ is in (1') in the *de re* supposition. On the other hand the constituent $[\lambda w \lambda t\ [{}^0B_{wt}\ {}^0KF_{wt}]]$ is in (2') in the *de dicto* supposition, there is no intensional descent here (applying to *w, t*), and the value of the proposition constructed by (2') in the actual possible world now does not depend on the value of the proposition P (in this actual world now) constructed by this component. In other words, Charles has an attitude to the whole proposition P, not to its (non-existing) value. Thus the proposition P' constructed by (2') can be true (if Charles is not properly informed about the French life and institutions). The constituent ${}^0KF$ is now in (2') also in the *de dicto* supposition, though there is the intensional descent here, for the value of P' in the actual world now does not depend on the non-existing value of the office KF in the actual world now. As Pavel Tichý says in [Tichý 1988, p. 217], *de dicto* is the dominant one of the two suppositions. Nevertheless, below we will make this Tichý's conception more precise.

We have shown that not only definite descriptions (or constructions of offices) can occur in the *de re / de dicto* supposition, but also the whole sentences (or constructions of propositions). We stated that in propositional attitudes what is believed, known, thought of, ... occurs in the *de dicto* supposition, and if a certain construction of the (believed) proposition occurs in the *de dicto* supposition, then also its components (particular subconstructions of offices or properties) are in the *de dicto* supposition, though they are applied to *w, t* (intensional descent) because they are in the *de dicto* context which is the dominant one.

But nowadays there is a lot of dispute about the so-called *de re attitudes*, see, e.g. [Kaplan 1969]. Thus Quine in [Quine 1990, p. 71] says:

> „... the propositional attitudes de re resist annexation to the scientific language, as propositional attitudes de dicto do not. At best the ascriptions de re are signals pointing a direction in which to look for informative ascriptions de dicto.

The reason for such a claim is according to Quine the fact that in propositional attitudes there is an opaque boundary between two ontologies, two worlds: the world of that individual to whom the attitude is ascribed and the world of that individual who is responsible for ascribing. This boundary-line is sometimes broken similarly as the actor sometimes forgets his/her role, but then we arrive at wrong conclusions. We will now show that *de re* attitudes can be precisely analysed when preserving this boundary, i.e. respecting, keeping separated the two perspectives: the perspective of the speaker (who is responsible for ascribing the attitude) and the perspective of the „believer“ to whom the attitude is ascribed.

Let us consider again a variant of the sentences (4) and (5) from the introductory *de re / de dicto* section F1, namely

(3) Charles believes that the president of USA is sick

(4) The president of USA is such that Charles believes him to be sick.

It is easy to verify that the two sentences are not equivalent and even particular inferences are not valid, i.e. (4) does not follow from (3), nor vice versa. The fact that (3) does not follow from (4) is obvious. Imagine a situation when Charles is not at the least acquainted with American life and institutions and being on a holiday he happens to meet Bill Clinton who travels incognito and falls sick. Then though it is true that Charles knows (and believes) that Bill Clinton is sick, the sentence (3) is not true. On the other hand we can claim „from the outside“ truly the sentence (4). It might, however, seem that the reverse implication holds, that (4) follows from (3). It is not the case. Imagine again a situation when our ignorant Charles does not have a slightest idea that Bill Clinton is the president of USA, but he has read in a newspaper that the president of USA suffers from a serious disease. Then the sentence (3) is certainly true but (4) is not. This claim is even more clear when we substitute the expression ‘the president of USA’ in both sentences by the expression ‘the king of France’. Then the sentence (3) can be true (it could not be, however, true, if we claimed that Charles *knows* ... [Duží 1999a]), but the sentence (4) does not have any truth-value.

As the reader certainly anticipates, the sentence (4) bears the existential commitment, and the principle of intersubstitutivity (*salva veritate*) holds in this case: when substituting another definite description that happens to have the same reference, as, e.g., ‘the husband of Hillary Clinton’ for ‘the president of USA’, the truth-value of the sentence is kept, or, more precisely, it follows from (4) (and from ‘the husband of Hillary Clinton is the president of USA’) that the husband of Hillary Clinton is such that Charles believes him to be sick. Hence in (3) ‘the president of USA’ occurs in the *de dicto* supposition, whereas in (4) in the *de re* supposition. The sentence (4) is an example of the *de re* attitude, and we will now analyse it. Let us still remark that in the literature we can also find the following formulation of the *de re* attitude:

Charles believes *of* the president of USA to be sick.

But since such a formulation could be considered to be only another way of expressing the sentence (3), (see [Tichý 1988, p. 217, 218]), we will better use the former formulation.

We first determine the type of particular entities about which these sentences speak, i.e. Ch(arles) / ι, P(resident) / $\iota_{\tau\omega}$, S(ick) / $(o\ \iota)_{\tau\omega}$, B(elieves) / $(o\ \iota\ o_{\tau\omega})_{\tau\omega}$, and we have:

(3') $\lambda w\lambda t\ [{}^0B_{wt}\ {}^0Ch\ [\lambda w\lambda t\ [{}^0S_{wt}\ {}^0P_{wt}]]\ ]$

de dicto

(4') $\lambda w\lambda t\ [\ \lambda x\ [{}^0B_{wt}\ {}^0Ch\ [\lambda w\lambda t\ [{}^0S_{wt}\ x]]\ ]\ {}^0P_{wt}\ ]$

de dicto

de re

The sentence (4) can be also read as „The president of USA belongs to the class of those whom Charles believes to be sick", which is just recorded by the construction (4'). We can see that ${}^0P_{wt}$ (the president of USA) is „removed from the context" of Charles's attitude, there is an intensional descent here and the truth-value of the proposition constructed by (4') in the actual world now depends on the value of the office constructed by ${}^0P$ in this world now, hence (Definition 13) ${}^0P$ (president ...) is in the *de re* supposition.

But now there is a question: What will happen when we perform an equivalent transformation of the construction (4') using β-reduction? We obtain

(4'') $\lambda w\lambda t\ [{}^0B_{wt}\ {}^0Ch\ [\lambda w\lambda t\ [{}^0S_{wt}\ {}^0P_{wt}]]\ ]$

which is identical with (3') where ${}^0P$ is in the *de dicto* supposition! Does it mean that the *de dicto* attitude in the end follows from the *de re* attitude? Or that the sentence (4) can be read in two ambivalent ways? It is not the case. The transformation from (4') to (4'') has not been performed correctly. Before applying the β-rule we have to rename (e.g. the inner) variables *w, t* (collision of variables!) in the „belief context". We thus get

(4''') $\lambda w\lambda t\ [\ \lambda x\ [{}^0B_{wt}\ {}^0Ch\ [\lambda w_1\lambda t_1\ [{}^0S_{w1t1}\ x]]\ ]\ {}^0P_{wt}\ ]$

and after applying the β-rule

(4'''') $\lambda w\lambda t\ [{}^0B_{wt}\ {}^0Ch\ [\lambda w_1\lambda t_1\ [{}^0S_{w1t1}\ {}^0P_{wt}]]\ ]$.

Now everything is all right: The subconstruction ${}^0P$ is in the construction (4'''') in the *de re* supposition, though it is „drawn into" the *de dicto* context which, though being dominant, does not affect the position of it, for the truth-value of the proposition constructed by (4'''') in a world *w* and time *t* depends on the value of the office P in *this w, t,* i.e. (Definition 13) ${}^0P$ occurs *de re*. In other worlds, variables w, t render the perspective of the speaker (who ascribes the attitude) and w_1, t_1 the perspective of the 'believer' (to whom the attitude is ascribed). These two perspectives („two worlds") are separated.

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